Solutions of quasi-linear wave equations polyhomogeneous at null infinity in high dimensions

Piotr T. Chruściel
Gravitational Physics
University of Vienna

Roger Tagne Wafo
Département de Mathématiques et Informatique
Faculté des Sciences, Université de Douala

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Abstract

We prove propagation of weighted Sobolev regularity for solutions of the hyperboloidal Cauchy problem for a class of quasi-linear symmetric hyperbolic systems, under structure conditions compatible with the Einstein-Maxwell equations in space-time dimensions $n+1 \geq 7$. Similarly we prove propagation of polyhomogeneity in dimensions $n+1 \geq 9$. As a byproduct we obtain, in those last dimensions, polyhomogeneity at null infinity of small data solutions of vacuum Einstein, or Einstein-Maxwell equations evolving out of initial data which are stationary outside of a ball.

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1 Introduction

A problem of current interest is the asymptotic behavior of solutions of hyperbolic equations in the radiation zone. For large (however, not for all) sets of initial data, this question can be reduced to one where the initial data are given on a Cauchy surface that resembles a hyperboloid in Minkowski space-time. In recent works [12, 13], polyhomogeneity of solutions of such Cauchy problems, with polyhomogeneous initial data, has been proved for a large class of semi-linear symmetric hyperbolic systems. The object of this work is to extend those results to quasi-linear equations satisfying certain structure conditions which are compatible with the vacuum Einstein equations, or with the Einstein-Maxwell equations, in space-time dimensions $n+1 \geq 9$.

A special case of our results is Theorem 7.2 below, where polyhomogeneity at null infinity of small data global solutions of the Einstein-Maxwell equations, evolving out of initial data which are stationary outside of a compact set, is established; this is perhaps the most significant result in this work. For clarity we repeat the relevant part of that theorem here:

THEOREM 1.1 *In dimensions $n + 1 \geq 9$ the global solutions of Einstein-Maxwell equations constructed in [18, 19] out from small initial data stationary outside of a compact set are polyhomogeneous at null infinity.*

The polyhomogeneous expansions above are in terms of powers of $\log r$ and negative integer powers of r in odd space dimension, while one has powers of $\log r$ and negative half-integer powers of r in even space dimension.

Theorem 1.1 should be compared with [4], where even space-time dimension $n + 1 \geq 6$ is assumed, where initial data *Schwarzschildian* outside of a compact set are considered, and where solutions which are smooth at null infinity are obtained. The methods of that last reference completely fail in odd space-time dimensions. Furthermore, in odd space dimensions, generic initial data which are only *stationary*, as opposed to *Schwarzschildian*, are likely to be polyhomogeneous, *but not smooth*, at null infinity, and generic such initial data are expected to be too singular to be covered by the approach in [4]. We also note the analysis in [3], which implies smoothness at null infinity of *exactly* stationary vacuum or electro-vacuum space-times, in even space-dimension, in space-time harmonic gauge. But the dimensions covered in [3] are precisely those not covered by the evolution theorems in [1, 4].

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2 Polyhomogeneity of solutions

2.1 Notation

The notation of [12] is used unless explicitly stated otherwise. However, to avoid a clash of notation with the symbol which is customarily used for the conformal factor arising in the rescaling of the metric, we will use the symbol $\mathscr{U}$ for the sets Ω of [12]:

$$\mathscr{U} = \left\{ (x, v^A, y) : \ 0 < x < y \,, \ v = (v^A) \in \mathscr{O} \,, \ 0 < y < 2T \right\} \,, \qquad (2.1)$$

where $\mathscr{O}$ is a compact manifold without boundary. We will write z for the joint set of variables (x, y, v^A).

Let W^α be a family of spaces, where α is a decay index, e.g. $W^\alpha = \mathscr{C}^\alpha_{\{x=0\},k}(\mathscr{U})$, or $W^\alpha = \mathscr{C}^\alpha_{\{0\le x\le y\},\infty}(\mathscr{U})$, etc. We define

$$W^{<\alpha} = \cap_{\sigma<\alpha} W^\sigma \,.$$

This notation is very useful to accommodate $\ln^n x$ factors that arise in the problem at hand: for example, in this notation we have

$$x^\alpha \ln^N x \in \mathscr{C}^{<\alpha}_{\{x=0\},\infty}(\mathscr{U}) \,.$$

We use a slight generalization of a definition of [13]: We shall say that a function $H(z, w)$ is $\mathscr{A}^\delta_{\{0\le x\le y\}}$-*polyhomogeneous in z with a uniform zero of order l in w* if the following hold: First, H is smooth in $w \in \mathbb{R}^N$ at fixed $z \in \mathscr{U}$. Next, it is required that for all $B \in \mathbb{R}$ and $k \in \mathbb{N}$ there exists a constant $\hat{C}(B)$ such that, for all $|w| \le B$ and $0 \le i \le \min(k, l)$,

$$\left\| \partial_w^i H(\cdot, w) \right\|_{\mathscr{C}^0_{\{0\le x\le y\},k-i}(\mathscr{U})} \le \hat{C}(B)|w|^{l-i} \,. \qquad (2.2)$$

Further,
$$\forall i \in \mathbb{N} \quad \partial_w^i H(\cdot, w) \in \mathscr{A}^\delta_{\{0\le x\le y\}} \qquad (2.3)$$

at fixed constant w. Finally we demand the uniform estimate for constant w's: $\forall \epsilon > 0, M \ge 0, i, k \in \mathbb{N} \ \exists\, C(\epsilon, M, i, k) \ \forall |w| \le M$ such that

$$\|\partial_w^i H(\cdot, w)\|_{\mathscr{C}^{-\epsilon}_{\{0\le x\le y\},k}(\mathscr{U})} \le C(\epsilon, M, i, k) \,. \qquad (2.4)$$

The qualification "in w" in "uniform zero of order l in w" will often be omitted. Similarly to [13], the small parameter ϵ has been introduced above to take into account the possible logarithmic blow-up of functions in $\mathscr{A}^\delta_{\{0\le x\le y\}}$

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at $x = 0$; for the applications to the nonlinear scalar wave equation or to the wave map equation on Minkowski space-time, the alternative simpler requirement would actually suffice: $\forall\, M \geq 0, i, k \in \mathbb{N} \;\; \exists\, C(M, i, k) \;\; \forall |w| \leq M$

$$\|\partial_w^i H(\cdot, w)\|_{\mathscr{C}^0_{\{0 \leq x \leq y\}, k}(\mathscr{U})} \leq C(M, i, k) \,, \tag{2.5}$$

again for constant w's. Functions which are smooth in (w, z), and have a zero of order l in w at $w = 0$, satisfy the above conditions.

2.2 The theorem

Let $\psi = (\psi_1, \psi_2)$ and set

$$f := (\psi, \varphi) \,, \quad \bar{f} := (\psi_1, x\psi_2, x\varphi) \,. \tag{2.6}$$

We shall say that a function G *satisfies the* NL-*condition* if there exist $N, p_i, q_i, m_i \in \mathbb{N}$ and functions H_i with a uniform zero of order m_i in the variable

$$
\begin{aligned}
w_i \;\; &:= \;\; x^{q_i \delta}(\bar{f}, x^2 \partial_x f, x^2 \partial_y f, x \partial_A f) \\
&\equiv \;\; x^{q_i \delta}(\psi_1, x\psi_2, x\varphi, x^2 \partial_x f, x^2 \partial_y f, x \partial_A f)
\end{aligned}
$$

such that

$$G = \sum_{i=1}^N x^{-p_i \delta} H_i(z, w_i) \,, \tag{2.7}$$

with, for $i = 1, \dots, N$,

$$m_i > \frac{p_i - \frac{1}{\delta}}{q_i} \,. \tag{2.8}$$

Our first main result is the following:

THEOREM 2.1 *Let $\mathscr{U}$ be defined in (2.1), suppose that $p \in \mathbb{Z}$, $q, 1/\delta \in \mathbb{N}^*$, $k \in \mathbb{N} \cup \{\infty\}$, and let*

$$\psi = (\psi_1, \psi_2)$$

and φ, with

$$\psi_1 \in \mathscr{C}^{<-1}_{\{0 \leq x \leq y\}, \infty} \cap \mathscr{C}^{<0}_{\{0 \leq x \leq y\}, 0} \,, \qquad \psi_2, \varphi \in \mathscr{C}^{<-1}_{\{0 \leq x \leq y\}, \infty} \,, \tag{2.9}$$

be a solution on $\mathscr{U}$ of the following system of equations:

$$
\begin{cases}
\partial_y \varphi + B_{\varphi\varphi}\varphi + B_{\varphi\psi}\psi = L_{\varphi\varphi}\varphi + L_{\varphi\psi}\psi + a + G_\varphi \\
\partial_x \psi + B_{\psi\varphi}\varphi + B_{\psi\psi}\psi = L_{\psi\varphi}\varphi + L_{\psi\psi}\psi + b + G_\psi
\end{cases}
\,, \tag{2.10}
$$

with the operators

$$L_{ij} = L_{ij}^A \partial_A + x L_{ij}^y \partial_y + x L_{ij}^x \partial_x \tag{2.11}$$

satisfying

$$L_{\varphi\varphi}^\mu \in x^\delta \mathscr{A}_{\{0 \le x \le y\}}^\delta \,, \quad L_{\psi\varphi}^\mu \,, L_{\varphi\psi}^\mu \,, L_{\psi\psi}^\mu \in \mathscr{A}_{\{0 \le x \le y\}}^\delta \tag{2.12}$$

(no symmetry hypotheses are made on the matrices L_{ij}^μ), while

$$B_{\varphi\varphi} \in C_\infty(\overline{\Omega}) + x^\delta \mathscr{A}_{\{0 \le x \le y\}}^\delta \,, \qquad B_{\varphi\psi}, B_{\psi\psi}, B_{\psi\varphi} \in \mathscr{A}_{\{0 \le x \le y\}}^\delta \tag{2.13a}$$

$$a, b \in x^{-1+\delta} \mathscr{A}_{\{0 \le x \le y\}}^\delta \,, \tag{2.13b}$$

$$\varphi|_{x=y} = \mathring{\varphi} \in x^{-1+\delta} \mathscr{A}_{\{x=0\}}^\delta \,, \ \psi|_{x=y} = \mathring{\psi} \in x^{-1+\delta} \mathscr{A}_{\{x=0\}}^\delta \,. \tag{2.14}$$

If the non-linear terms G_φ, G_ψ satisfy the NL-condition, then

$$(\psi, \varphi) \in \mathscr{A}_{\{0 \le x \le y\}}^\delta \times x^{\delta-1} \mathscr{A}_{\{0 \le x \le y\}}^\delta \,;$$

more precisely

$$\psi \in x^\delta \mathscr{A}_{\{0 \le x \le y\}}^\delta + \mathscr{A}_{\{y=0\}}^\delta \,, \tag{2.15a}$$

$$\varphi \in x^{\delta-1} \mathscr{A}_{\{x=0\}}^\delta + x^{\delta-1} y \mathscr{A}_{\{0 \le x \le y\}}^\delta \,. \tag{2.15b}$$

In particular for any $\tau > 0$ we have

$$(\psi, \varphi)|_{\{y \ge \tau\}} \in \mathscr{A}_{\{x=0\}}^\delta \times x^{\delta-1} \mathscr{A}_{\{x=0\}}^\delta \,,$$

which shows that the solution is polyhomogeneous with respect to $\{x = 0\}$ on $\{y \ge \tau\}$.

PROOF: This theorem is a generalization of the semi-linear case, Theorem 3.7 of [12], and can be proved by following step by step the proof given there. A detailed exposition can be found in [25]. $\qquad\square$

3   Propagation of polyhomogeneity for the Einstein-Maxwell equations

Let us show that Theorem 2.1 applies to the source-free Einstein-Maxwell equations; we will make extensive appeal to [4]. More generally, consider a system of second order wave equations of the form

$$\eta^{\alpha\beta} \frac{\partial^2 f}{\partial x^\alpha \partial x^\beta} = -H^{\alpha\beta}(x^\mu, f, \partial f, \partial\partial f) \frac{\partial^2 f}{\partial x^\alpha \partial x^\beta} + F(f, \partial f, x^\mu) \,, \tag{3.1}$$

for a map f with values in $\mathbb{R}^N$ for some N, where η is the $(n+1)$–dimensional Minkowski metric. (The map f in this section should *not* be confused with the map f appearing in (2.6), compare (3.18) below.) The Einstein-Maxwell equations in the harmonic-Lorenz gauge can be written in this form, with $f := (g_{\mu\nu} - \eta_{\mu\nu}, A_\mu)$, then

$$H^{\mu\nu} := g^{\mu\nu} - \eta^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}(g_{\alpha\beta} - \eta_{\alpha\beta}) + \text{ quadratic terms}$$

depends only upon $g_{\mu\nu} - \eta_{\mu\nu}$, while F is a quadratic form in ∂f with coefficients depending upon $g_{\mu\nu} - \eta_{\mu\nu}$. Thus, in the Einstein-Maxwell case the source function F has a uniform zero of order two, while the functions $H^{\mu\nu}$ all have a uniform zero of order one.

As in [4], and similarly[1] to [13], we use a mapping $\phi : x \mapsto y$ from the of the future timelike cone with vertex 0, $I^+_{\eta,x}(0)$, of a Minkowski space-time, which we denote $(\mathbb{R}^{n+1}_x, \eta_x)$, into the past timelike cone with vertex 0 of another Minkowski space-time, $(\mathbb{R}^{n+1}_y, \eta_y)$, defined by

$$\phi : I^+_{\eta,x}(0) \to \mathbb{R}^{n+1}_y \quad \text{by} \quad x^\alpha \mapsto y^\alpha := \frac{x^\alpha}{\eta_{\lambda\mu}x^\lambda x^\mu} \ . \tag{3.2}$$

It is easy to check that ϕ is a bijection from $I^+_{\eta,x}(0)$ onto $I^-_{y,\eta}(0)$, with inverse

$$\phi^{-1} : y^\alpha \mapsto x^\alpha \quad \text{by} \quad x^\alpha := \frac{y^\alpha}{\eta_{\lambda\mu}y^\lambda y^\mu} \ . \tag{3.3}$$

Moreover ϕ is a conformal mapping between Minkowski metrics:

$$\eta_{\alpha\beta}dx^\alpha dx^\beta = \Omega^{-2}\eta_{\alpha\beta}dy^\alpha dy^\beta \ , \tag{3.4}$$

where Ω is a function defined on all $\mathbb{R}^{n+1}_y$, given by

$$\Omega := -\eta_{\alpha\beta}y^\alpha y^\beta \ . \tag{3.5}$$

We work within $I^-_{y,\eta}(0)$ and to the future of a hypersurface

$$\mathscr{S}_{\tau_0} := \{y^0 = \tau_0\} \ , \quad \tau_0 < 0 \ ,$$

where we set

$$\rho \equiv |\vec{y}| := \sqrt{\sum_{i=1}^n (y^i)^2} \ , \quad x := -|\vec{y}| - y^0 \geq 0 \ , \quad y := y^0 - |\vec{y}| + 1 \geq 0 \ ,$$

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[1] In [4] one works within $I^+_{\eta,x}(0)$, while in [13] the complement of $I^+_{\eta,x}(0)$ is considered. However, the methods of [13] apply to both situations.

so that

$$\Omega = -x(1-y) \,, \quad \partial_x = -\frac{1}{2}\Big(\sum_{i=1}^{n}\frac{y^i}{|\vec{y}|}\frac{\partial}{\partial y^i}+\frac{\partial}{\partial y^0}\Big) \,, \quad \partial_y = -\frac{1}{2}\Big(\sum_{i=1}^{n}\frac{y^i}{|\vec{y}|}\frac{\partial}{\partial y^i}-\frac{\partial}{\partial y^0}\Big) \,,$$

and

$$y^\alpha \frac{\partial}{\partial y^\alpha} = (y-1)\partial_y + x\partial_x \,.$$

Furthermore the flat d'Alembertian $\Box_{\eta,y}$ associated with the coordinates y^μ equals

$$\Box_{\eta,y} = 4\partial_x\partial_y - \frac{2(n-1)}{1-x-y}\Big(\partial_x + \partial_y\Big) + \frac{4\Delta_h}{(1-x-y)^2} \,,$$

where Δ_h is the canonical Laplacian on S^{n-1}.

It should be kept in mind that we are interested in x small and y bounded away from one.

The general relation between the wave operator on scalar functions in two conformal metrics transforms the left-hand-side of (3.1) into the following partial differential operator

$$\eta^{\alpha\beta}\frac{\partial^2(\Omega^{-\frac{n-1}{2}} f \circ \phi^{-1})}{\partial y^\alpha \partial y^\beta} \equiv \Omega^{-\frac{n+3}{2}}(\eta^{\alpha\beta}\frac{\partial^2 f}{\partial x^\alpha \partial x^\beta}) \circ \phi^{-1} \,. \tag{3.6}$$

We introduce the following new set of scalar functions on $\mathbb{R}^{n+1}_y$

$$\hat{f} := \Omega^{-\frac{n-1}{2}} f \circ \phi^{-1} \,, \tag{3.7}$$

so that the system (3.1) reads

$$\eta^{\alpha\beta}\frac{\partial^2 \hat{f}}{\partial y^\alpha \partial y^\beta} = -\Omega^{-\frac{n+3}{2}}\Big\{H^{\alpha\beta}(x,f,\partial f,\partial\partial f)\frac{\partial^2 f}{\partial x^\alpha \partial x^\beta} - F(x,f,\partial f)\Big\} \circ \phi^{-1} \,, \tag{3.8}$$

and we need to analyse the structure of the right-hand side. As calculated in detail in [4], if we set

$$A^\alpha_\mu := \frac{\partial y^\alpha}{\partial x^\mu} \circ \phi^{-1} \equiv -\Omega\delta^\alpha_\mu - 2y^\alpha \eta_{\mu\beta}y^\beta \,, \tag{3.9}$$

which is bounded on any bounded set of $\mathbb{R}^{n+1}_y$, we can write

$$\frac{\partial f}{\partial x^\mu} \circ \phi^{-1} = A^\alpha_\mu \frac{\partial(f \circ \phi^{-1})}{\partial y^\alpha} = \Big(-x(1-y)\frac{\partial}{\partial y^\mu} - 2\eta_{\mu\alpha}y^\alpha\big((y-1)\frac{\partial}{\partial y}+x\frac{\partial}{\partial x}\big)\Big)f\circ\phi^{-1} \,. \tag{3.10}$$

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(We emphasise the occurrence of a factor of x in front of each derivative except ∂_y; that will play an important role in what follows.) Similarly

$$\frac{\partial^2 f}{\partial x^\lambda \partial x^\mu} \circ \phi^{-1} = \left\{ x^2(1-y)^2 \frac{\partial^2}{\partial y^\lambda \partial y^\mu} + 4x(1-y)\eta_{\alpha(\lambda}y^\alpha \frac{\partial}{\partial y^{\mu)}} (x\partial_x + (y-1)\partial_y) \right.$$
$$+ 4\eta_{\lambda\alpha}\eta_{\mu\beta}y^\alpha y^\beta ((y-1)\partial_y + x\partial_x)^2$$
$$\left. + \left[4\eta_{\lambda\alpha}\eta_{\mu\beta}y^\alpha y^\beta + 2x(1-y)\eta_{\lambda\mu} \right] ((y-1)\partial_y + x\partial_x) \right\} f \circ \phi^{-1} . \quad (3.11)$$

If we now set $f \circ \phi^{-1} = \Omega^{\frac{n-1}{2}} \hat{f}$, we find:

$$\frac{\partial (f \circ \phi^{-1})}{\partial y^\alpha} \equiv \frac{\partial (\Omega^{\frac{n-1}{2}} \hat{f})}{\partial y^\alpha} = \Omega^{\frac{n-1}{2}} \frac{\partial \hat{f}}{\partial y^\alpha} - (n-1)\Omega^{(n-3)/2}y_\alpha \hat{f} , \quad (3.12)$$

and

$$\frac{\partial^2 (f \circ \phi^{-1})}{\partial y^\alpha \partial y^\beta} \equiv (x(1-y))^{\frac{n-1}{2}} \frac{\partial^2 \hat{f}}{\partial y^\alpha \partial y^\beta} - (n-1)(x(1-y))^{(n-3)/2} \left(\eta_{\beta\sigma}y^\sigma \frac{\partial \hat{f}}{\partial y^\alpha} + \eta_{\alpha\sigma}y^\sigma \frac{\partial \hat{f}}{\partial y^\beta} \right)$$
$$+ \frac{(n-1)}{2}(x(1-y))^{\frac{n-5}{2}} D_{\alpha\beta}\hat{f} , \quad (3.13)$$

with

$$D_{\alpha\beta} := 2(n-3)\eta_{\lambda\alpha}\eta_{\mu\beta}y^\lambda y^\mu + 2(x(y-1))\eta_{\alpha\beta} . \quad (3.14)$$

Collecting all this, we conclude that

$$\frac{\partial^2 f}{\partial x^\lambda \partial x^\mu} \circ \phi^{-1} = (x(1-y))^{\frac{n-1}{2}} \left\{ x^2(1-y)^2 \frac{\partial^2}{\partial y^\lambda \partial y^\mu} + 4x(1-y)\eta_{\alpha(\lambda}y^\alpha \frac{\partial}{\partial y^{\mu)}} (x\partial_x + (y-1)\partial_y) \right.$$
$$+ 4\eta_{\lambda\alpha}\eta_{\mu\beta}y^\alpha y^\beta ((y-1)\partial_y + x\partial_x)^2 + 2(n-1)x(1-y)y^\alpha \eta_{\alpha(\lambda} \frac{\partial}{\partial y^{\mu)}}$$
$$+ \left[4n\eta_{\lambda\alpha}\eta_{\mu\beta}y^\alpha y^\beta + 2x(1-y)\eta_{\lambda\mu} \right] ((y-1)\partial_y + x\partial_x)$$
$$\left. + (n-1)\left[(n+1)\eta_{\alpha\mu}\eta_{\lambda\beta}y^\alpha y^\beta + x(1-y)\eta_{\lambda\mu} \right] \right\} \hat{f} . \quad (3.15)$$

The second term on the right-hand side of (3.8) is

$$\Omega^{-\frac{n+3}{2}} F\left(f, \frac{\partial f}{\partial x} \right) \circ \phi^{-1} = \Omega^{-\frac{n+3}{2}} F\left(\Omega^{\frac{n-1}{2}} \hat{f}, \Omega^{\frac{n-1}{2}} A_\mu^\alpha \left(\frac{\partial \hat{f}}{\partial y^\alpha} - (n-1)\Omega^{-1} y_\alpha \hat{f} \right) \right) . \quad (3.16)$$

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Now, $A^\alpha_\mu y_\alpha = \Omega y_\mu$, and it follows from (3.10) that the right-hand side of the last equation can be rewritten as

$$\left(x(1-y)\right)^{-\frac{n+3}{2}} \times$$
$$F\left(\left(x(1-y)\right)^{\frac{n-1}{2}}\hat{f}, \left(x(1-y)\right)^{\frac{n-1}{2}}\left(x(y-1)\frac{\partial\hat{f}}{\partial y^\mu} - 2\eta_{\mu\alpha}y^\alpha((y-1)\partial_y\hat{f} + x\ \partial_x\hat{f}) - (n-1)y_\mu\hat{f}\right)\right).$$
(3.17)

As shown in [12, 13], the left-hand-side of (3.1) can be brought to the form needed in Theorem 2.1 by setting

$$\psi_1 = \hat{f}, \quad \psi_2 = (\partial_y\hat{f}, \partial_A\hat{f}), \quad \varphi = \partial_x\hat{f}. \tag{3.18}$$

Here $\partial_A f = \partial_{v^A}f$, where the v^A's are local coordinates on the sphere. To bring (3.17) to the desired form (2.7), the choice

$$p_2\delta = \frac{n+3}{2}, \quad q_2\delta = \frac{n-3}{2},$$

provides the supplementary power of x needed in the arguments of F to satisfy the structure conditions of Theorem 2.1, provided that we choose $1/(2\delta) \in \mathbb{N}^*$ in even space-dimensions; any $1/\delta \in \mathbb{N}^*$ is admissible in odd ones. If we assume that F has a uniform zero of order m_2, condition (2.8) will now be satisfied for

$$m_2 > \frac{n+1}{n-3} = 1 + \frac{4}{n-3} \iff n \geq 4 \text{ and } m_2 \geq \begin{cases} 6, & n = 4; \\ 4, & n = 5; \\ 3, & n = 6,7; \\ 2, & n \geq 8. \end{cases}$$
(3.19)

(In the Einstein-Maxwell case we have $m_2 = 2$, which enforces $n \geq 8$.)

Let us turn our attention to the first term at the right-hand side of (3.1). In what follows we will consider the following restricted class of non-linearities: we assume that, after replacing f by $\Omega^{\frac{n-1}{2}}\hat{f}$ and changing variables $x^\mu \to y^\mu$ as above, the terms $H^{\alpha\beta}$ takes the form

$$H^{\alpha\beta} = G^{\alpha\beta}(\Omega^{\frac{n-1}{2}}\hat{f}, \Omega^{\frac{n-1}{2}+1}\partial_{y^\mu}\hat{f}, \Omega^{\frac{n-1}{2}+2}\partial_{y^\nu}\partial_{y^\rho}\hat{f}), \tag{3.20}$$

with a uniform zero of order m_0. Such a structure will clearly be obtained from a function in (3.1) which depends only upon f, in particular this will be the case for the Einstein or the Einstein-Maxwell equations, with $m_0 = 1$.

Using (3.15) we can write

$$\Omega^{-\frac{n+3}{2}} H^{\mu\nu} \partial_{x^\mu} \partial_{x^\nu} f = x^{-\frac{n+3}{2}+\frac{n-1}{2}} F_1(H^{\alpha\beta}, \hat{f}, \partial_{y^\alpha}\partial_{y^\beta}\hat{f}, \partial_{y^\gamma}\hat{f})$$

where F_1 is linear in the second, third, and fourth argument. Assuming (3.20), this can be rewritten as

$$
\begin{aligned}
\Omega^{-\frac{n+3}{2}} H^{\mu\nu} \partial_{x^\mu} \partial_{x^\nu} f &= x^{-\frac{n+7}{2}} F_1(H, x^{\frac{n-1}{2}+2}\hat{f}, x^{\frac{n-1}{2}+2}\partial_{y^\alpha}\partial_{y^\beta}\hat{f}, x^{\frac{n-1}{2}+2}\partial_{y^\gamma}\hat{f}) \\
&= x^{-\frac{n+7}{2}} F_2(x^{\frac{n-1}{2}}\hat{f}, x^{\frac{n-1}{2}+2}\partial_{y^\alpha}\partial_{y^\beta}\hat{f}, x^{\frac{n-1}{2}+1}\partial_{y^\gamma}\hat{f}) \,,
\end{aligned}
$$

where F_2 has a uniform zero of order $m_1 = m_0 + 1$. With the restrictions on δ as before, we will obtain the right structure by setting

$$p_1\delta = \frac{n+7}{2}\,, \qquad q_1\delta = \frac{n-1}{2}\,,$$

and the NL-condition will hold provided that $m_1 := m_0 + 1$ satisfies

$$m_1 > \frac{n+5}{n-1} = 1 + \frac{6}{n-1}\,, \qquad \Longleftrightarrow \qquad m_0 \geq \begin{cases} 7, & n = 2; \\ 4, & n = 3; \\ 3, & n = 4; \\ 2, & n = 5,6,7; \\ 1, & n \geq 8. \end{cases} \qquad (3.21)$$

In particular the structure conditions will be satisfied by the Einstein-Maxwell equations in space-dimensions larger than or equal to eight.

The hypothesis (3.20) will not be satisfied in general if $H^{\mu\nu}$ in (3.1) is a non-linear function of f and $\partial_{x^\mu} f$, for then H will belong instead to the following class of functions (compare (3.16))

$$H = G(\Omega^{\frac{n-1}{2}}\hat{f}, \Omega^{\frac{n-1}{2}}\partial_{y^\mu}\hat{f}, \Omega^{\frac{n-1}{2}+1}\partial_{y^\nu}\partial_{y^\rho}\hat{f})\,, \qquad (3.22)$$

An analysis similar to the one above shows that, for $H^{\mu\nu}$'s which are a finite sum of terms of the form (3.22), we will obtain the right structure by setting

$$p_1\delta = \frac{n+5}{2}\,, \qquad q_1\delta = \frac{n-3}{2}\,,$$

and the NL-condition will hold provided that $m_1 = m_0 + 1$ satisfies

$$m_1 > \frac{n+3}{n-3} = 1 + \frac{6}{n-3} \qquad \Longleftrightarrow \qquad n \geq 4 \text{ and } m_0 \geq \begin{cases} 7, & n = 4; \\ 4, & n = 5,; \\ 3, & n = 6; \\ 2, & n = 7,8,9; \\ 1, & n \geq 10. \end{cases}$$

$$(3.23)$$

The reader should have no troubles similarly working out the conditions on the nonlinearity for general H's which depend on f, $\partial_{x^\mu} f$ and $\partial_{x^\mu} \partial_{x^\nu} f$.

Summarizing, we have proved:

THEOREM 3.1 *Let f be a solution of equation (3.1), define ψ_1, ψ_2, and φ by (3.18), where $\hat{f}$ is given by (3.7). Suppose that (3.19) holds, and assume that either (3.20) with (3.21) hold, or (3.22) with (3.23) hold. If (2.9) and (2.14) hold, then the conclusions of Theorem 2.1 apply. In particular Theorem 2.1 applies to the Einstein-Maxwell equations in space-time dimensions $n+1 \geq 9$.*

4   Towards solutions with a polyhomogeneous Scri

In order to establish existence of solutions of the vacuum Einstein equations, in sufficiently high dimensions, with a polyhomogeneous Scri, it remains to construct appropriate initial data, and show that the corresponding solutions are in the right function spaces.

Recall, now, that large classes of polyhomogeneous hyperboloidal initial data have been constructed in [2] (the emphasis in that reference is on $n = 3$ at several places, but the general results there show that the conformal method, starting from smooth or polyhomogeneous seed fields, provides polyhomogeneous solutions of the general relativistic vacuum constraint equations in any dimension $n \geq 3$). There is little doubt that large collections of initial data so constructed provide polyhomogeneous data for the harmonically reduced equations of the last section, but we have not checked this in detail. Instead, we will follow the standard-by-now strategy of using initial data which are stationary outside of a compact set. So, in Section 4.2, we provide large classes of Corvino-Schoen type initial data with polyhomogeneous asymptotics on hyperboloids. One of the reasons for proceeding this way is that small such initial data lead to global, geodesically complete solutions [19, 20].

One then needs to verify that the associated solutions satisfy the space-time weighted regularity conditions needed in Theorem 2.1. One could hope that the Lindblad-Rodnianski type estimates of Loizelet [19, 20] would provide that information. It turns out that the available estimates, for space-times obtained by evolving small initial data of Section 4.2, are not sufficient for our polyhomogeneity result; this is analyzed in Section 4.3. This means that the desired estimates have to be derived from scratch, which will be done in the remainder of this paper.

4.1 Stationary vacuum metrics in higher dimensions

The only way, so far, of obtaining space-times with controlled asymptotic behavior near i^0 is to use initial data sets which are stationary at large distances. We will outline the construction of such data in Section 4.2, but before doing this it is convenient to start with a short discussion of stationary metrics in higher dimensions; our presentation follows [3].

Consider a vacuum Lorentzian metric ^{n+1}g in any space-time-dimension $n \geq 3$, with Killing vector $X = \partial/\partial t$. In the region where X is timelike there exist adapted coordinates in which ^{n+1}g takes the form

$$^{n+1}g = -V^2(dt + \underbrace{\theta_i dx^i}_{=\theta})^2 + \underbrace{g_{ij}dx^i dx^j}_{=g} , \qquad (4.1)$$

$$\partial_t V = \partial_t \theta = \partial_t g = 0 . \qquad (4.2)$$

The vacuum Einstein equations (with vanishing cosmological constant) read (see, e.g., [15])

$$\begin{cases} V\nabla^*\nabla V = \frac{1}{4}|\lambda|_g^2 , \\ \mathrm{Ric}(g) - V^{-1}\mathrm{Hess}\,_g V = \frac{1}{2V^2}\lambda \circ \lambda , \\ \mathrm{div}(V\lambda) = 0 , \end{cases} \qquad (4.3)$$

where

$$\lambda_{ij} = -V^2(\partial_i \theta_j - \partial_j \theta_i) , \quad (\lambda \circ \lambda)_{ij} = \lambda_i{}^k \lambda_{kj} .$$

We assume that there exists $\alpha > 0$ such that

$$g_{ij} - \delta_{ij} = O(r^{-\alpha}) , \ \partial_k g_{ij} = O(r^{-\alpha-1}) , \qquad (4.4)$$

similarly for $V - 1$ and θ_i. A redefinition $t \to t + \psi$, introduces a gauge transformation

$$\theta \to \theta + d\psi ,$$

and one can exploit this freedom to impose restrictions on θ. For our purposes it is convenient to impose the *harmonic gauge*, $\Box t = 0$, which reads

$$\partial_i(\sqrt{\det g}Vg^{ij}\theta_j) = 0 . \qquad (4.5)$$

Equation (4.5) can always be achieved by replacing θ by $\theta + d\psi$, and solving the resulting linear equation for ψ, *cf., e.g.,* [5, **?**] for the relevant isomorphism theorems.) One can then introduce new coordinates [**?**] which are harmonic for g.

In space-harmonic coordinates, and in the gauge (4.5), the system (4.3) is elliptic, and standard considerations show that the functions g_{ij}, V and θ_i have a polyhomogeneous expansion in terms of $\log r$ and inverse powers of r. Furthermore, $^{n+1}g_{\mu\nu}$ is Schwarzschild in the leading order, and there exist constants α_{ij} such that

$$\theta_i = \frac{\alpha_{ij}x^j}{r^n} + O(r^{-n}) \ .$$

It is of interest to enquire whether or not the logarithmic powers are essential in the polyhomogeneous expansion. It has long been know in space-dimension three that, for metrics which are stationary and vacuum in the asymptotic region, coordinate systems *exist* where no $\log r$ terms arise whenever the ADM mass is non-zero [22]. The same property is true for *static* solutions with non-zero ADM mass in space-dimension four [3]. Now, in the evolution theorems used below we need all coordinates to satisfy the wave equation,

$$\Box x^\mu = 0 \ , \tag{4.6}$$

and the transition from the coordinates used in [3] to the coordinates satisfying (4.6) might introduce log terms: This is exactly what happens for the Schwarzschild metric in $n = 4$, which *does have* a logarithmic term in its asymptotic expansion in a natural choice of wave coordinates [4], but this is the only dimension where this happens for Schwarzschild.

In general, (4.6) is achieved by changing space-coordinates $x^i \to x^i + \psi^i(x^j)$ (recall that t is already harmonic), thus solving a linear equation for ψ^i; by standard results (see, e.g., [6]) the ψ^i's will have a full asymptotic expansion in terms of powers of $\ln r$ and inverse powers of r, and so will the space-time metric in the new coordinate system, when transformed from the space-harmonic ones. In view of the calculations in [4], this implies the existence of polyhomogeneous asymptotics of the initial data on hyperboloids at $\mathscr{I}$, as needed in Theorem 2.1.

Rather surprisingly, in even space-dimensions larger than or equal to six the space-coordinates used in [3] satisfy (4.6), and so does the time coordinate. It follows that the analysis of stationary solutions in [3] directly provides wave coordinates in which no log terms occur in those dimensions.

4.2 Corvino-Schoen data in higher dimensions

So far we have considered metrics which are exactly stationary. Now, there exists a construction due to Corvino and Schoen [?, ?] (see also [9, 10], and

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also the more recent Reference [8], where the construction is carried out under considerably weaker asymptotic conditions) which allows one to glue exactly stationary ends to asymptotically Euclidean initial data sets. Some details of this construction have been presented in those references in dimension three only, but the construction generalises to any dimension, as follows: Recall that the construction requires a family of stationary reference metrics which cover the whole range of asymptotic charges. In dimension $3 + 1$ this is provided by the family of metrics obtained by boosting and translating the Kerr metrics. In higher dimensions one such family can be obtained by boosting and translating the Myers-Perry metrics [21]. Note that the question, whether or not the reference solutions have naked singularities is irrelevant for the problem at hand because here one only needs the solutions at large distances. (Similarly to the Kerr family, all the metrics in the family so obtained have a timelike ADM momentum, and therefore can only be glued to asymptotically flat initial data which also have this property; this is no restriction for well behaved initial data sets which are spin, or for space-dimensions up to seven, and is expected not to be a restriction for well behaved initial data sets in general, but this has not been proved at the moment of writing of this work.)

So let R_x, ϵ_k be positive constants and consider the collection, say $\mathbf{C}_{R_x, \epsilon_k}$ of general relativistic electro-vacuum initial data sets $(\mathbb{R}^n, g, K)$ which are stationary outside a coordinate ball $B(R_x)$ and with weighted Sobolev norm controlling k-derivatives of the metric smaller than ϵ_k. Here k should be sufficiently large as in [4, 20], and the norm should be the one described in those references. From what has been said this collection is non-empty, and contains an open set (in the topology associated to the norm) around Minkowski space-time.

Now, for the Schwarzschild metric in dimension $n + 1$ with $n \geq 4$, and in harmonic coordinates, the boundary of the domain of influence of a ball is sandwiched between two hypersurfaces $t - r = \text{const}$ [4, Section 5.3]. This remains true for stationary electro-vacuum metrics because the leading order behaviour of the metric coincides with the Schwarzschild one (compare [7, Appendix A]). This implies that the maximal globally hyperbolic development of all initial data in $\mathbf{C}_{R_x, \epsilon_k}$ contains hyperboloidal hypersurfaces, the asymptotic region of which is contained in that part of the space-time where the metric is stationary. So our considerations of the previous section apply to this region, leading to polyhomogeneous initial data on such hypersurfaces. Since the leading order deviation of the metric from the flat one is Schwarzschildian, the tensor field $\hat{h} := \Omega^{-\frac{n-1}{2}}(g - \eta)$, that plays a key role

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in our analysis, is $O(x^{(n-2)-(n-1)/2}) = O(x^{(n-3)/2})$, and in fact

$$\hat{h} \in x^{(n-3)/2}(\mathscr{A}^\delta_{\{x=0\}} \cap L^\infty)\,, \quad \partial_{y^0}\hat{h} \in x^{(n-5)/2}(\mathscr{A}^\delta_{\{x=0\}} \cap L^\infty)\,, \qquad (4.7)$$

with $\delta = 1$ on any hyperboloid whose asymptotic part is contained in the stationary region.

4.3   Lindblad-Rodnianski-Loizelet metrics near $\mathscr{I}$

In this section we analyze how the asymptotic behavior of the small-data space-times constructed in [19] (compare [16, 17]) relates to the differentiability conditions needed in Theorem 2.1. We find that sharper decay rates along outgoing null geodesics would be needed for a direct proof of polyhomogeneity using our approach. The estimates established here are then combined with the results of our analysis in subsequent sections to provide a rather more involved proof of polyhomogeneity.

We start by recalling some notation of [16, 17, 19]. Let $\mathscr{Z}$ denote the following set of vectors on Minkowski space-time:

$$\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha},\ \alpha = 0, 1, \ldots, n;$$

$$Z_{\alpha\beta} = x_\alpha \partial_\beta - x_\beta \partial_\alpha,\ \alpha, \beta = 0, 1, \ldots, n;$$

$$Z_0 = \sum_{\alpha=0}^{n} x^\alpha \partial_\alpha = t\partial_t + \sum_{i=1}^{n} x_i \partial_i = t\partial_t + r\partial_r\,.$$

Here, as usual, $x_0 = -x^0 = -t$, $x_i = x^i$ for $i = 1\ldots, n$. Let the spherical coordinates (r, θ^A) be defined as

$$\begin{cases} t = x^0\,, \\ r = \big(\sum_{i=1}^{n}(x^i)^2\big)^{1/2}\,, \\ x^i = r\omega^i(\theta^A),\ i = 1, \ldots, n, \end{cases} \qquad (4.8)$$

where θ^A denotes any local coordinates on the sphere S^{n-1}. The vector fields

$$L = \partial_t + \partial_r = \partial_t + \omega^i \partial_i\,, \qquad \underline{L} = \partial_t - \partial_r = \partial_t - \omega^i \partial_i\,.$$

are tangent, respectively transverse, to the light cones $t - r = \text{const}$. We note

$$Z_0 = t\partial_t + r\partial_r\,.$$

Furthermore, the Z_{ij}'s, $i, j = 1 \ldots, n$ are tangent to the spheres $S^{n-1} \subset \mathbb{R}^n$, and can be purely expressed in terms of the θ^A's.

Let $T \geq 0$, set $T^\mu = (T, 0, \ldots, 0)$, in this section it is more convenient to consider instead the following variation of (3.2):

$$y^\mu = \frac{x^\mu + T^\mu}{(x^\alpha + T^\alpha)(x_\alpha + T_\alpha)} \iff x^\mu + T^\mu = \frac{y^\mu}{y^\alpha y_\alpha} . \qquad (4.9)$$

This provides a conformal transformation from the future causal cone centred at T^μ in the Minkowski space-time with coordinates x^μ to the past causal cone of the origin in the Minkowski space-times with coordinates y^μ, and with conformal factor $\Omega = y^\alpha y_\alpha = \frac{1}{-(t+T)^2 + r^2}$.

To make contact with Section 2 we set

$$x = -y^0 - \rho, \quad y = y^0 - \rho + 1 \quad \text{where} \quad \rho = \Big(\sum_{i=1}^{n} (y^i)^2 \Big)^{1/2} ,$$

so that

$$\begin{cases} y^0 = \frac{1}{2}(y - x - 1) \\ \rho = \frac{1}{2}(-y - x + 1) \\ y^i = \frac{1}{2}(-y - x + 1)\omega^i(v^A), \ i = 1, \ldots, n \end{cases} . \qquad (4.10)$$

Here ω^i is a unit vector, and the v^A's denote local coordinates on S^{n-1} in the y–coordinates. One can take $\omega^i(\theta^A) = \omega^i(v^A)$, $i = 1, \ldots, n$; we will make this choice, and simply write ω^i in both x^μ and y^μ coordinates.

Letting $\mathscr{H}_s$ be the following family of hyperboloids,

$$\mathscr{H}_s = \Big\{ x^0 - s = \sqrt{s^2 + r^2} \Big\} , s > 0 ,$$

we will have

$$\phi(\mathscr{H}_s) = \{ y^0 = -\frac{1}{2s} \}$$

in particular $\phi(\mathscr{H}_1) = \{ y^0 = -\frac{1}{2} \}$.

The methods of Section 2 involve the vector fields

$$x\partial_x, \ y\partial_y, \ \partial_A = \frac{\partial}{\partial v^A}, \ A = 1, \ldots n - 1.$$

By straightforward calculations one finds, keeping in mind that $\rho = \frac{r}{(t+T)^2 - r^2}$ for $t + T \geq r$,

$$x = \frac{1}{t + T + r}, \quad 1 - y = \frac{1}{t + T - r} \iff r = \frac{1}{2x} - \frac{1}{2(1 - y)}, \quad t = \frac{1}{2x} + \frac{1}{2(1 - y)} - T$$

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$$\begin{cases} x\partial_x = -\frac{1}{2}(t+T+r)(\partial_t + \partial_r) \,, \\ (1-y)\partial_y = \frac{1}{2}(t+T-r)(\partial_t - \partial_r) \,, \\ \partial_A = \text{linear combinations of } Z_{ij}, \ i,j = 1,\dots,n \ . \end{cases} \qquad (4.11)$$

The coefficients in the equation for ∂_A above depend only upon the angular variables, and a finite number of coordinate patches v^A can be chosen so that in each of those patches the coefficients are uniformly bounded together with derivatives of any order.

This leads us to

PROPOSITION 4.1 *Let $T, T_0 > 0$, $t \geq 0$ and suppose that*

$$1 - T \leq t - r \leq T_0 \iff 0 \leq y \leq 1 - \frac{1}{T+T_0} \,. \qquad (4.12)$$

For all $k \in \mathbb{N}$, $\forall \, (i,j,\gamma) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}^{n-1}$ satisfying $i+j+|\gamma| \leq k$, and for any function $f \in C^k$ we have

$$[x\partial_x]^i \partial_y^j \partial_v^\gamma f = \sum_{|I| \leq k, \, Z \in \mathcal{Z}} H_I^{ij\gamma}(\theta, y) Z^I f \qquad (4.13)$$

with $|H_I^{ij\gamma}(\theta, y)| \leq C(i, j, I, T, T_0)$.

PROOF: Using (4.11) one can rewrite $x\partial_x$ and ∂_y as

$$x\partial_x = -\frac{1}{2}(Z_0 - \omega^i Z_{0i} + T(\partial_t + \omega^i \partial_i)) \,, \qquad (4.14)$$

$$\partial_y = \underbrace{\frac{1}{2(1-y)}}_{=:\varphi_1(y)} \underbrace{(Z_0 + \omega^i Z_{0i} + T(\partial_t - \omega^i \partial_i))}_{=:\tilde{Z}} \,. \qquad (4.15)$$

It is thus clear that $x\partial_x$, and any of its powers, have the right structure. Next, the factor $\varphi_1(y)$ appearing in (4.15) is bounded on any compact subinterval of $[0,1)$ (note that $y = 1$ corresponds to the tip of the past causal cone centred at the origin of the y^μ-coordinates). One easily finds by induction that

$$\partial_y^j = \sum_{i=1}^j \varphi_i(y) \tilde{Z}^i \,,$$

where the functions φ_i are bounded on compact subsets of $[0,1)$, whence the result. $\qquad \square$

We wish to obtain the asymptotic behavior of the fields occurring in Theorem 2.1 for the global solutions

$$f := (h_{\mu\nu}, A_\mu)$$

of the Einstein-Maxwell equations constructed in [19]. In order to apply Theorem 2.1 we need

$$\psi_1 = \hat{f} \in \mathscr{C}^{<0}_{\{0\leq x\leq y\},0}; \ \left(\psi_2 = (\partial_y \hat{f}, \partial_A \hat{f}), \ \varphi = \partial_x \hat{f}\right) \in \mathscr{C}^{<-1}_{\{0\leq x\leq y\},\infty} \ ,$$

where

$$\hat{f} = \Omega^{-\frac{n-1}{2}} f \circ \phi^{-1}.$$

Now,

$$\Omega = -x(1-y)$$

which implies that for any $\alpha \in \mathbb{R}$ we have

$$(x\partial_x)^i (\Omega^\alpha f) = \Omega^\alpha \sum_{j=0}^{i} C(\alpha, i, j)(x\partial_x)^j f \ . \tag{4.16}$$

Similarly,

$$(y\partial_y)^i (\Omega^\alpha f) = \Omega^\alpha \sum_{j=0}^{i} C'(\alpha, i, j, x, y)(y\partial_y)^j f \ , \qquad \partial_y^i (\Omega^\alpha f) = \Omega^\alpha \sum_{j=0}^{i} C''(\alpha, i, j, x, y)\partial_y^j f \ , \tag{4.17}$$

where the functions C' and C'' are bounded for x in, say, $[0, x_0]$, and for y bounded away from 1.

The solutions constructed in [19] satisfy the following: there exists $0 < \delta < 1/4$ such that for $t \geq 0$ and $|t - r| \leq C_1$, and for all I there exists a constant C, depending upon I and C_1, such that

$$|Z^I f(t, x^i)| \leq C(1 + t + r)^{\frac{1-n}{2} + \delta} \ , \tag{4.18}$$

$$|\bar{\partial} Z^I f(t, x^i)| \leq C(1 + t + r)^{\frac{-1-n}{2} + \delta} \ , \tag{4.19}$$

where

$$\bar{\partial} \in \left\{\partial_t + \partial_r, \ r^{-1}\partial_A\right\} = \left\{-2x^2\partial_x, \frac{2x(1-y)}{1-x-y}\partial_A\right\} \ . \tag{4.20}$$

Now,

$$\frac{1+t+r}{t+T+r} = 1 + \frac{1+T}{t+T+r} \in \left[1, \frac{1+T}{T}\right] \quad \text{for} \quad T > 0, \ t \geq 0 \ ,$$

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so (4.18)-(4.19) imply

$$|Z^I f(t, x^i)| \leq Cx^{\frac{n-1}{2}-\delta}, \quad |\bar{\partial} Z^I f(t, x^i)| \leq Cx^{\frac{n-1}{2}+1-\delta}. \quad (4.21)$$

From (4.16)-(4.17) and Proposition 4.1 we obtain

$$[x\partial_x]^i \partial_y^j \partial_v^\gamma \hat{f}(x, y, v)$$
$$= [x\partial_x]^i \partial_y^j \partial_v^\gamma \Omega^{\frac{1-n}{2}} f(t, x^i)$$
$$= \Omega^{\frac{1-n}{2}} \sum_{0 \leq m \leq i} \sum_{0 \leq \ell \leq j} c(i, j, m, \ell, n, x, y)[x\partial_x]^m \partial_y^\ell \partial_v^\gamma f(t, x^i)$$
$$= \Omega^{\frac{1-n}{2}} \sum_{0 \leq m \leq i} \sum_{0 \leq \ell \leq j} c(i, j, m, \ell, n, x, y) \sum_{|I| \leq k,\ Z \in \mathcal{Z}} H_I^{m\ell\gamma}(\theta, y) Z^I f(t, x^i).$$

Using the first inequality in (4.21) we conclude that for any $0 < \epsilon \leq 1$ and for $0 \leq y \leq 1 - \epsilon$ we have

$$\left| [x\partial_x]^i [y\partial_y]^j \partial_v^\gamma \hat{f}(x, y, v) \right| \leq \left| [x\partial_x]^i \partial_y^j \partial_v^\gamma \hat{f}(x, y, v) \right| \leq Cx^{-\delta},$$

while it should be clear from (4.20) that the second inequality in (4.21) does not provide any new information in the coordinate ranges assumed above. In any case the property

$$\left(\psi_1 = \hat{f}, \psi_2 = (\partial_y \hat{f}, \partial_A \hat{f}) \right) \in \mathscr{C}_{\{0 \leq x \leq y\},0}^{-\delta}, \quad \varphi = \partial_x \hat{f} \in \mathscr{C}_{\{0 \leq x \leq y\},\infty}^{-1-\delta} \quad (4.22)$$

immediately follows. Unfortunately, to apply Theorem 2.1 one would need δ to be an arbitrary positive number, while in (4.22) δ is a small number determined by the initial data. So, as already pointed out, we need to derive the necessary estimates by different methods. This is the purpose of the sections that follow.

5 Weighted energy estimates near a null boundary

Let $(\mathscr{M}, \mathfrak{g})$ be an $(n + 1)$-dimensional space-time. We consider systems of quasi-linear of nonlinear wave equations, with diagonal principal part of the form

$$\Box_{\mathfrak{g}} u = F(.\ , u, \partial u), \quad (5.1)$$

on a neighborhood of a null hypersurface of $\mathscr{M}$. We suppose that the background metric $\mathfrak{g}$ is a smooth function of the coordinates, of the unknown vector valued function u, as well as its first order derivatives.

All calculations below will be done for a real valued function u, the result for a vector valued function is obtained by summing over the components.

5.1 The hypotheses, and the geometry of the problem

5.1.1 The hypotheses

We will consider the Cauchy problem associated to equation (5.1), the initial data will be given on a hypersurface $\mathscr{S}_0$. We will evolve these initial data to obtain a solution of our problem in a *past* one-sided neighborhood of a null hypersurface

$$\mathscr{N} = \{x = 0\}$$

forming the boundary, or a subset thereof, of the domain of dependence of $\mathscr{S}_0$. Here, and throughout, x stands for a positive function such that dx has no zeros on $\{x = 0\}$. We will be working in a neighborhood of $\{x = 0\}$, chosen so that x is a coordinate there, of the form

$$\mathscr{V} \equiv [\tau_0,\, \tau_1\,[\, \times\,]\,0, x_0\,[\, \times\, \mathscr{O}\,,$$

where $[\tau_0,\, \tau_1\,[$ corresponds to the time interval, $]\,0, x_0\,[$ the range of the variable x, and $\mathscr{O}$ is an $(n-1)$-dimensional compact submanifold of $\mathscr{M}$ without boundary. The coordinates will be denoted by (τ, x, v), with $v = (v^A)_{A=1}^{n-1}$ the coordinates on $\mathscr{O}$. We assume that ∂_τ is timelike, and we choose the time-orientation on $\mathscr{M}$ such that the vector ∂_τ is everywhere future directed.

One can think of the set $\mathscr{U}$ of (2.1) as a subset of the coordinate patch above, compare Figure 6.2, page 78.

On the components of the metric $\mathfrak{g}$ with respect to the coordinates (τ, x, v), we assume the following:

1. We suppose that

$$\exists \epsilon_0 > 0\,, \quad \text{such that} \quad -\mathfrak{g}^{\tau\tau} \geq \epsilon_0 \tag{5.2}$$

   everywhere on $\mathscr{V}$.

2. The components $\mathfrak{g}^{\tau\tau}$ and $\mathfrak{g}^{\tau x}$ can be written as

$$\mathfrak{g}^{\tau\tau} = -1 + x\mathfrak{h}^0(\tau, x, v^A) \quad \text{and} \quad \mathfrak{g}^{\tau\tau} + \mathfrak{g}^{\tau x} = x\mathfrak{h}^1(\tau, x, v^A) \tag{5.3}$$

   where the functions $\mathfrak{h}^0$ and $\mathfrak{h}^1$ are bounded on bounded sets.

3. On the components $\mathfrak{g}^{xA}$ and $\mathfrak{g}^{xx}$ we assume that

$$\mathfrak{g}^{xA} = \mathrm{O}(x) \quad \text{and} \quad \mathfrak{g}^{\tau\tau} + 2\mathfrak{g}^{\tau x} + \mathfrak{g}^{xx} = 1 + \mathrm{O}(x) \tag{5.4}$$

   and we set $\mathfrak{g}^{xA} = x\mathfrak{h}^A$ and $\mathfrak{g}^{\tau\tau} + 2\mathfrak{g}^{\tau x} + \mathfrak{g}^{xx} = 1 + x\mathfrak{h}$, where $\mathfrak{h}$ and $\mathfrak{h}^A$ are bounded functions on bounded sets. We further suppose that

$$\mathfrak{g}^{\tau\tau} + 2\mathfrak{g}^{\tau x} + \mathfrak{g}^{xx} > 0\,.$$

4. The vector field

$$Y^\nu \partial_\nu := \partial_\tau - \partial_x \tag{5.5}$$

is assumed to be everywhere timelike on $\mathscr{V}$ and future directed. This vector will be used to contract the energy momentum tensor.

The set of functions $(\mathfrak{h}, \mathfrak{h}^\mu)$ will be denoted by $\mathfrak{h}^\sharp$ and $\mathfrak{g}^\sharp$ will denote the inverse matrix of the matrix $(\mathfrak{g}_{\mu\nu})$.

REMARK 5.1 It follows from the above that the vector ∇x (where ∇ is the covariant derivative compatible with the metric $\mathfrak{g}$) can be decomposed as

$$\nabla x = \omega^{(1)} + \beta(x)\omega^{(2)} \tag{5.6}$$

where $\omega^{(1)}$ is causal future directed, and that there exists a constant C_0 such that

$$|\beta(x)| \leq C_0 x , \quad |\omega^{(2)}| \leq C_0 |\mathfrak{h}^\sharp| . \tag{5.7}$$

EXAMPLE 5.2 As an example, consider a conformally rescaled asymptotically flat solution of asymptotically vacuum Einstein equations in *Bondi coordinates* near Scri [23], with the metric taking the form

$$\tilde{\mathfrak{g}}_B = e^{2\beta} dx \mathring{\otimes} dy + \chi dy \mathring{\otimes} dy + 2\gamma \mathring{\otimes} dy + \mu , \tag{5.8}$$

for some functions β and χ, and a one-form field γ. (Here y corresponds to the Bondi retarded time u, and $x = 1/2r$ is half the inverse of the luminosity distance r. E.g., for the Minkowski metric in any dimensions, $\beta = \chi = 0 = \gamma$.) In $3+1$ dimensions, for smoothly compactifiable metrics, the Einstein equations imply, for matter fields decaying sufficiently fast, that $\beta = O(x^2)$ as well as

$$\chi = O(x^2) , \quad \gamma_A = O(x^2) , \tag{5.9}$$

with derivatives behaving in the obvious way. Equation (5.9) remains valid for asymptotically vacuum metrics which, after conformal rescaling, are polyhomogeneous and C^1 (see [14, Section 6] or [11, Appendix C.1.2]), while for general $\mathscr{A}^\delta_{\{x=0\}} \cap L^\infty$–polyhomogeneous asymptotically vacuum metrics one has [14, Equations (2.15)-(2.19) with $H = X^a = 0$] the asymptotic behaviors $\beta = O(x^2 \ln^N x)$ and

$$\chi = O(x^2) , \quad \gamma_A = O(x^2 \ln^N x) , \tag{5.10}$$

for some N. Here "asymptotically vacuum" requires, for polyhomogeneous metrics, that the components of the energy-momentum tensor in asymptotically Minkowskian coordinates satisfy (see [14, end of Section 2])

$$T_{\mu\nu} = o(r^{-2}) . \tag{5.11}$$

We have
$$\det \mathfrak{g} = -\frac{1}{4} \det \mu \, ,$$

which, for a Lorentzian metric, shows that μ must be a non-degenerate $(n-1) \times (n-1)$ tensor field. It is simple to check that the inverse metric $\mathfrak{g}^\sharp = \mathfrak{g}^{\alpha\gamma} \partial_\alpha \mathring{\otimes} \partial_\gamma$ is given by the formula

$$
\begin{aligned}
\mathfrak{g}^\sharp &= 4(-\chi + |\gamma|_\mu^2)\partial_x \mathring{\otimes} \partial_x + 4\partial_x \mathring{\otimes} \partial_y - 4\gamma^\sharp \mathring{\otimes} \partial_x + \mu^\sharp \\
&= 4\partial_x \mathring{\otimes} \left(\partial_y + (-\chi + |\gamma|_\mu^2)\partial_x - \gamma^\sharp \right) + \mu^\sharp \, ,
\end{aligned}
\tag{5.12}
$$

with $\mu^\sharp = \mu^{AB} \partial_A \mathring{\otimes} \partial_B$, where μ^{AB} is the matrix inverse to μ_{AB}, $\gamma^\sharp = \mu^{AB}\gamma_A \partial_B$, $|\gamma|_\mu^2 = \mu^\sharp(\gamma, \gamma) = \mu^{AB}\gamma_A\gamma_B$, and $\mathring{\otimes}$ denotes the symmetric tensor product. We note
$$\mathfrak{g}(\nabla y, \nabla y) = \mathfrak{g}^{yy} = 0 \, ,$$

which makes clear the null character of the level sets of y, and implies, by a well-known argument, that the integral curves of

$$\nabla y = \mathfrak{g}^{\alpha\gamma} \partial_\alpha y \partial_\gamma = \mathfrak{g}^{y\gamma} \partial_\gamma = 2\partial_x$$

are null geodesics.

Consider a new coordinate system (x, v^A, τ), where

$$(x, y) \longrightarrow (x, \tau = \frac{y - x}{2}) \, , \tag{5.13}$$

so that

$$\partial_x \longrightarrow \partial_x - \frac{1}{2}\partial_\tau \, , \quad \partial_y = \frac{1}{2}\partial_\tau \, . \tag{5.14}$$

Thus

$$\mathfrak{g}^\sharp = 4(-\chi + |\gamma|_\mu^2)(\partial_x - \frac{1}{2}\partial_\tau)\mathring{\otimes}(\partial_x - \frac{1}{2}\partial_\tau) + 4(\partial_x - \frac{1}{2}\partial_\tau)\mathring{\otimes}(\frac{1}{2}\partial_\tau) - 4\gamma^\sharp\mathring{\otimes}(\partial_x - \frac{1}{2}\partial_\tau) + \mu^\sharp \, ,$$

giving

$$\mathfrak{g}^{xx} = 4(-\chi + |\gamma|_\mu^2) \, , \quad \mathfrak{g}^{x\tau} = 1 - 2(-\chi + |\gamma|_\mu^2) \, , \quad \mathfrak{g}^{xA} = -2\mu^{AB}\gamma_B \, , \tag{5.15}$$
$$\mathfrak{g}^{\tau A} = \mu^{AB}\gamma_B \, , \quad \mathfrak{g}^{\tau\tau} = -1 + (-\chi + |\gamma|_\mu^2) \, , \quad \mathfrak{g}^{AB} = \mu^{AB} \, . \tag{5.16}$$

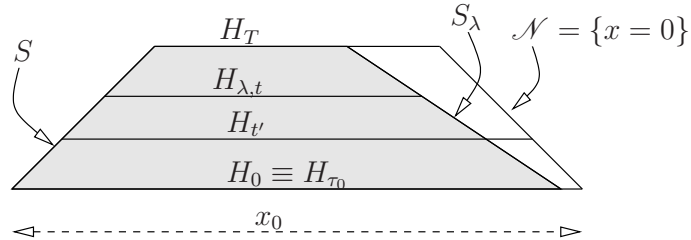This, together with (5.10), shows that (5.3)-(5.4) hold for such metrics.

Figure 5.1: The sets $\mathscr{U}_{\lambda,T}$ (shaded) and $\mathscr{U}_T$ (the outermost trapezium). In this picture (but *not* in our hypotheses) the light-cones have forty-five degrees slopes, as in Minkowski space-time.

5.1.2 The slices

In this section we describe the sets within which we obtain our estimates, see Figure 5.1. Let $t \in [\tau_0, 0[$ run over the range of the time coordinate τ of the previous section.

- Let $\lambda \in [0,1]$ parameterize a family of spacelike hypersurfaces S_λ, which approach $\{x = 0\}$ when λ approaches zero, of the form

$$S_\lambda = \{(\tau, x, v^A) : \ x = \sigma_\lambda(\tau)\} \,,$$

where σ_λ is a C^1function such that:

  - $\sigma_0(\tau) \equiv 0$ i.e. $S_0 = \{x = 0\}$
  - S_λ is everywhere spacelike.

One can legitimately raise concerns about existence of the family S_λ with global behaviour as above when the space-time under consideration is being constructed as a solution of a Cauchy problem. While the aim of this work is to prove that the resulting space-time will have properties as in Figure 5.1, this is not known a priori. Now, one way to proceed is to construct the solution as the limit of solutions of linear equations on a sequence of metrics, each of those metrics satisfying controlled weighted energy estimates as proved below. In particular each space-time in this sequence is globally hyperbolic, with the set $\{x = 0\}$ being part of the boundary of the domain of dependence of the initial surface. For each metric in the sequence a relevant family

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S_λ can be constructed using e.g. Cauchy time functions; no details will be given as no significant difficulties are involved. This can then be used to justify our estimates for each metric in the sequence, and for the solution.

- By S we denote a smooth spacelike hypersurface transverse to $\{\tau = \tau_0\}$ defined by

$$S = \{(\tau, x, v^A) : \ x = \sigma(\tau)\} \,, \tag{5.17}$$

where σ is a smooth function of τ such that

$$0 < \sigma(\tau_1) \le \sigma(\tau) \le \sigma(\tau_0) = x_0 \,.$$

- $\mathbf{H}_{\lambda,t} = \{(\tau, x, v^A); \ \tau = t, \ \sigma_\lambda(x) \le x \le \sigma(\tau)\}, \ \mathscr{U}_{\lambda,T} = \underset{\tau_0 \le t \le T}{\cup} \mathbf{H}_{\lambda,t}.$

- $\mathbf{H}_t = \{(\tau, x, v^A); \ \tau = t, \ 0 \le x \le \sigma(\tau)\}, \quad \mathscr{U}_T = \underset{\tau_0 \le t \le T}{\cup} \mathbf{H}_t.$

Note that the boundary $\partial\mathscr{U}_{\lambda,t}$ of the region $\mathscr{U}_{\lambda,t}$ is made of four pieces, S_λ, S, $\mathbf{H}_{\lambda,\tau_0}$ and $\mathbf{H}_{\lambda,t}$. We recall that, for $\theta \in \mathbb{R}$, $j \in \mathbb{N}$ the spaces $\mathscr{C}_j^\theta(\mathbf{H}_{\lambda,\tau})$, $\mathscr{B}_j^\theta(\mathbf{H}_{\lambda,\tau})$, $\mathscr{H}_j^\theta(\mathbf{H}_{\lambda,\tau})$ and $\mathscr{G}_j^\theta(\mathbf{H}_{\lambda,\tau})$ are defined in the appendix of [13].

5.1.3 The causality properties of the boundary

We want to show that under the assumptions we made on certain components of the metric, all the hypersurfaces defined above have the nature which will be needed when applying the Stokes' theorem or when we will like to use the positivity of the stress energy momentum tensor.

The vector $\nabla\tau = \nabla^\mu(\tau)\partial_\mu = \mathfrak{g}^{\mu\nu}\delta_\nu^\tau\partial_\mu = \mathfrak{g}^{\tau\tau}\partial_\tau + \mathfrak{g}^{x\tau}\partial_x + \mathfrak{g}^{A\tau}\partial_A$ is normal to the hypersurfaces $\mathbf{H}_t$ and $\mathbf{H}_{\lambda,t}$, and the square of its norm is $\mathfrak{g}(\nabla\tau, \nabla\tau) = \mathfrak{g}^{\tau\tau} < 0$. Therefore $\nabla\tau$ is time-like and thus these hypersurfaces are space-like. Their *past directed* unit normal is

$$\eta = \eta^\mu\partial_\mu = \frac{1}{\sqrt{|\mathfrak{g}^{\tau\tau}|}}(\mathfrak{g}^{\tau\tau}\partial_\tau + \mathfrak{g}^{x\tau}\partial_x + \mathfrak{g}^{A\tau}\partial_A) \,. \tag{5.18}$$

We also note de following

$$\eta_\mu = \mathfrak{g}_{\mu\nu}\eta^\nu = \frac{1}{\sqrt{|\mathfrak{g}^{\tau\tau}|}}\mathfrak{g}_{\mu\nu}\mathfrak{g}^{\nu\tau} = \frac{1}{\sqrt{|\mathfrak{g}^{\tau\tau}|}}\delta_\mu^\tau$$

that is

$$\eta_\mu dx^\mu = \frac{1}{\sqrt{|\mathfrak{g}^{\tau\tau}|}}d\tau \,. \tag{5.19}$$

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As far as the hypersurfaces S_λ are concerned, the functions σ_λ are assumed to be such that the normal $N = \nabla\{-x + \sigma_\lambda(\tau)\}$ is timelike and the outward unit normal to this hypersurface is such that the integral of the contracted energy momentum tensor is negative (see (5.37)). The same remark holds for the hypersurface S.

5.2 Estimates on the space derivatives of the solution

We want to derive weighted energy inequalities for solutions of (5.1). These inequalities will be used to prove existence of a solution satisfying the hypothesis of the theorem of polyhomogeneous solution of quasi-linear wave equation near scri.

5.2.1 The stress energy momentum tensor and its properties

The stress-energy tensor of the system (5.1) is given by

$$T_{\mu\nu} := \nabla_\mu u \nabla_\nu u - \frac{1}{2}\mathfrak{g}_{\mu\nu}\nabla^\alpha u \nabla_\alpha u \ .$$

The explicit form of $T_0{}^0$, (the component of the tensor T which in general determines the energy density of the system) in local coordinates system is given by:

$$
\begin{aligned}
T_0{}^0 &= \nabla^0 u \nabla_0 u - \frac{1}{2}\nabla^\alpha u \nabla_\alpha u \\
&= \mathfrak{g}^{0\beta}\nabla_\beta u \nabla_0 u - \frac{1}{2}\mathfrak{g}^{\alpha\beta}\nabla_\alpha u \nabla_\beta u \\
&= \left\{\mathfrak{g}^{00}\nabla_0 u \nabla_0 u + \mathfrak{g}^{0i}\nabla_i u \nabla_0 u\right\} - \frac{1}{2}\left\{\mathfrak{g}^{00}\nabla_0 u \nabla_0 u + 2\mathfrak{g}^{0i}\nabla_0 u \nabla_i u + \mathfrak{g}^{ij}\nabla_i u \nabla_j u\right\} \\
&= \frac{1}{2}\left\{\mathfrak{g}^{00}(\nabla_0 u)^2 - \mathfrak{g}^{ij}\nabla_i u \nabla_j u\right\} = -\frac{1}{2}\left\{-\mathfrak{g}^{00}(\nabla_0 u)^2 + |Du|^2\right\} \quad (5.20)
\end{aligned}
$$

with $|Du|^2 := \mathfrak{g}^{ij}\nabla_i u \nabla_j u$.

The tensor T is symmetric and its divergence is given by

$$
\begin{aligned}
\nabla_\mu T_\nu{}^\mu &= \Box_\mathfrak{g} u \nabla_\nu u \\
&= F\nabla_\nu u \qquad \text{when } u \text{ solves (5.1)} \ . \qquad (5.21)
\end{aligned}
$$

Further, one of the useful properties of the tensor T is its positivity: For any vectors fields v^α and w^α both causal future-pointing we have:

$$T_\nu{}^\mu v^\nu w_\mu \geq 0 \ . \qquad (5.22)$$

REMARK 5.3 In the particular frame (τ, x, v^A) we will be interested with, let us calculate the quantity $T^Y := T(\partial_\tau - \partial_x, d\tau) = T_\tau{}^\tau - T_x{}^\tau$ which we will use as energy density. From (5.20) we have:

$$T_\tau{}^\tau = \frac{1}{2}\left\{\mathfrak{g}^{\tau\tau}\left(\partial_\tau u\right)^2 - \mathfrak{g}^{xx}\left(\partial_x u\right)^2 - 2\mathfrak{g}^{xA}\partial_x u\partial_A u - \mathfrak{g}^{AB}\partial_A u\partial_B u\right\} \ .$$

This expression shows that in the case we are concerned with, $T_\tau{}^\tau$ cannot be used to control the energy of the system near $\{x = 0\}$ since the metric component $\mathfrak{g}^{xx}$ can degenerate there. On the other hand we have

$$T_x{}^\tau = \mathfrak{g}^{\tau\tau}\partial_\tau u\partial_x u + \mathfrak{g}^{\tau x}\left(\partial_x u\right)^2 + \mathfrak{g}^{\tau A}\partial_x u\partial_A u \ ,$$

therefore we deduce the following expression of T^Y :

$$T^Y = \frac{1}{2}\Big\{\mathfrak{g}^{\tau\tau}\left(\partial_\tau u\right)^2 \quad - \quad 2\mathfrak{g}^{\tau\tau}\partial_\tau u\partial_x u - \left(\mathfrak{g}^{xx} + 2\mathfrak{g}^{\tau x}\right)\left(\partial_x u\right)^2$$
$$-2\left(\mathfrak{g}^{xA} + \mathfrak{g}^{\tau A}\right)\partial_x u\partial_A u - \mathfrak{g}^{AB}\partial_A u\partial_B u\Big\} \ . \ (5.23)$$

Now, if we set

$$\begin{cases} \lambda = \mathfrak{g}^{\tau\tau} + \mathfrak{g}^{xx} + 2\mathfrak{g}^{x\tau} = 1 + O(x) > 0 \quad \text{(by hypothesis)} \\ \xi^A = \mathfrak{g}^{xA} + \mathfrak{g}^{A\tau} \\ \kappa^{AB} = \frac{\xi^A\xi^B}{\lambda} \end{cases} \ ,$$

then we obtain the following decomposition of T^Y

$$T^Y = -\frac{1}{2}\left\{-\mathfrak{g}^{\tau\tau}\left(\partial_\tau u - \partial_x u\right)^2 + \lambda\left(\partial_x u + \frac{\left(\mathfrak{g}^{xA} + \mathfrak{g}^{A\tau}\right)}{\lambda}\partial_A u\right)^2 + \left(\mathfrak{g}^{AB} - \kappa^{AB}\right)\partial_A u\partial_B u\right\} \ .$$
$$(5.24)$$

The above decomposition shows that the quantity T^Y controls uniformly the energy of the system if and only if there exists $\epsilon_0 > 0$ (which can be made to coincide with the one occurring in (5.2)) such that

$$\lambda > \epsilon_0, \text{ and } \left(\mathfrak{g}^{AB} - \kappa^{AB}\right)\zeta_A\zeta_B \geq \epsilon_0 \sum_A (\zeta_A)^2 \ ; \qquad (5.25)$$

the existence of such a constant follows already from our previous hypotheses. It turns out that if we have a priori bounds on the L^∞ norms of $\mathfrak{g}^\sharp$ from above and below, this expression can be used to control all the components of the stress energy tensor. In fact we have

$$|T_\nu{}^\mu| = |\mathfrak{g}^{\mu\sigma}\partial_\sigma u\partial_\nu u - \frac{1}{2}\delta_\nu{}^\mu\mathfrak{g}^{\alpha\beta}\partial_\alpha u\partial_\beta u| \leq C|\mathfrak{g}^\sharp||\partial u|^2 \leq C|\mathfrak{g}^\sharp||T_\tau{}^\tau - T_x{}^\tau| \ ;$$
$$(5.26)$$

here the constant C depends upon ϵ_0, and is allowed to change after each inequality symbol in general.

REMARK 5.4 For further purposes we note that, using the vector field $\partial_\tau - \partial_x$, the principal part of the d'Alembertian has the following form:

$$
\begin{aligned}
\mathfrak{g}^{\alpha\beta}\partial_{\alpha\beta} &= \mathfrak{g}^{\tau\tau}(\partial_\tau - \partial_x)^2 + 2\left(\mathfrak{g}^{\tau\tau} + \mathfrak{g}^{\tau x}\right)(\partial_\tau - \partial_x)\partial_x + 2\mathfrak{g}^{\tau A}(\partial_\tau - \partial_x)\partial_A \\
&\quad + \left(\mathfrak{g}^{\tau\tau} + 2\mathfrak{g}^{\tau x} + \mathfrak{g}^{xx}\right)\partial_x^2 + 2\left(\mathfrak{g}^{xA} + \mathfrak{g}^{\tau A}\right)\partial_x\partial_A + \mathfrak{g}^{AB}\partial_A\partial_B \ .
\end{aligned}
$$

$$(5.27)$$

5.2.2 Estimates on the first derivatives of the solution

We want to derive some energy inequalities for the solution u of the system (5.1). For this purpose, we consider the weighted energy at an instant t of the evolution of the system defined using the vector field $\partial_\tau - \partial_x$; recall $T^Y = T_\tau{}^\tau - T_x{}^\tau$:

$$
E[u(t)] = -\int_{\mathbf{H}_t} x^{-2\alpha} T^Y \frac{dx}{x} d^{n-1}\nu_{t,x}
$$

$$(5.28)$$

where $d^{n-1}\nu_{t,x}$ is the measure defined on $\{t\} \times \{x\} \times \mathscr{O}$ by the metric $\mathfrak{g}$ (as will be made precise shortly), and $\alpha \le 0$ a real parameter the range of which will be given later. We set

$$
E_\lambda[u(t)] = -\int_{\mathbf{H}_{\lambda,t}} x^{-2\alpha} T^Y \frac{dx}{x} d^{n-1}\nu_{t,x} \ .
$$

$$(5.29)$$

Our strategy will be to obtain a bound of $E[u(t)]$ from an uniform bound (with respect to λ) of $E_\lambda[u(t)]$. We will apply the divergence theorem to the energy-momentum tensor; this holds e.g. for $C^{1,1}_{\mathrm{loc}}$ functions u (first derivatives locally Lipschitz continuous). We want to establish the following (recall that ϵ_0 is the constant arising in (5.2) and in (5.25), while C_0 is defined in (5.7)):

PROPOSITION 5.5 *Let $\alpha \le -\frac{1}{2}$. Under hypotheses (5.2)-(5.4) and (5.25), there exists a constant C_1 depending upon ϵ_0, C_0, α such that for all*

$$
\tau \in [\tau_0, \tau_1] \quad and \quad u \in C^{1,1}_{\mathrm{loc}}
$$

satisfying (5.1), we have

$$
\begin{aligned}
E_\lambda[u(\tau)] &\le C_1 \Bigg\{ E_\lambda[u(\tau_0)] + \int_{\tau_0}^\tau \Big\{ \|F(s)\|^2_{\mathscr{H}^\alpha_0(\mathbf{H}_{\lambda,s})} + \Big(1 + \|\mathfrak{h}^\sharp\|_{L^\infty} + \|\mathfrak{g}^\sharp\|_{L^\infty}\Big) \\
&\quad \times \Big(1 + \|\mathfrak{g}\|^2_{L^\infty(\mathbf{H}_{\lambda,s})} + \|\mathfrak{g}^\sharp\|^2_{L^\infty(\mathbf{H}_{\lambda,s})} + \|(\partial_\tau - \partial_x)\mathfrak{g}^\sharp\|^2_{L^\infty(\mathbf{H}_{\lambda,s})}\Big) E_\lambda[u(s)] \Big\} ds \Bigg\}
\end{aligned}
$$

$$(5.30)$$

PROOF: Stokes' theorem for the vector field $\Lambda^\mu = x^{-2\alpha-1}T_\nu{}^\mu Y^\nu$ on $\mathscr{U}_{\lambda,\tau}$ (compare Fig. 5.1) gives

$$\int_{\partial\mathscr{U}_{\lambda,\tau}} x^{-2\alpha-1}T_\nu{}^\mu Y^\nu \eta_\mu dS = \int_{\mathscr{U}_{\lambda,\tau}} \nabla_\mu\{x^{-2\alpha-1}T_\nu{}^\mu Y^\nu\}dV \qquad (5.31)$$

for an arbitrary differentiable vector field Y. Here

$$dV = \sqrt{|\det \mathfrak{g}|}d\tau \wedge dx \wedge d^{n-1}v \ , \qquad (5.32)$$

where $\det \mathfrak{g}$ is the determinant of the metric $\mathfrak{g}$. Further, on non-characteristic parts of the boundary, η_μ is the unit outwards pointing conormal, and

$$dS = \sqrt{|\det \gamma|}d^n y \ , \qquad (5.33)$$

with y^i, $i = 1,\ldots,n$, a system of coordinates on the corresponding boundary, and γ the metric induced on it by the metric $\mathfrak{g}$; i.e. $\gamma = j^*\mathfrak{g}$, j being the canonical injection of the boundary into the manifold. (On characteristic parts of the boundary, a convenient choice of η_μ and dS will be made as need arises). In the case under consideration, $\partial\mathscr{U}_{\lambda,\tau}$ is made of four pieces $\mathbf{H}_{\lambda,\tau_0}$, $\mathbf{H}_{\lambda,\tau}$, together with

$$S_{\lambda,\tau} := S_\lambda \cap \{0 \le t \le \tau\} \text{ and } S^\tau := S \cap \{0 \le t \le \tau\} \ .$$

Therefore the identity (5.31) reads:

$$\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1}T_\nu{}^\mu Y^\nu \eta_\mu dS \ + \ \int_{\mathbf{H}_{\lambda,\tau_0}} x^{-2\alpha-1}T_\nu{}^\mu Y^\nu \eta_\mu dS + \int_{S_{\lambda,\tau}} x^{-2\alpha-1}T_\nu{}^\mu Y^\nu \eta_\mu dS$$

$$+ \ \int_{S^\tau} x^{-2\alpha-1}T_\nu{}^\mu Y^\nu \eta_\mu dS = \int_{\mathscr{U}_{\lambda,\tau}} \nabla_\mu\{x^{-2\alpha-1}T_\nu{}^\mu Y^\nu\}dV \ .$$

$$(5.34)$$

The left-hand-side of equation (5.34) is made of four terms which will be labeled in their order of appearance L_1, L_2, L_3 and L_4. As mentioned before, we choose the vector field $Y = Y^\mu \partial_\mu$ to be equal to $\partial_\tau - \partial_x$. Once this choice is made, let us look at each of the terms $L_i, i = 1, 2, 3, 4$. Recall that (see equation (5.19)) on $\mathbf{H}_{\lambda,\tau}$ we have:

$$\eta_\mu dx^\mu = \frac{1}{\sqrt{|\mathfrak{g}^{\tau\tau}|}}d\tau \quad \text{which implies that} \quad T_\nu{}^\mu Y^\nu \eta_\mu = \frac{1}{\sqrt{|\mathfrak{g}^{\tau\tau}|}}\{T_\tau{}^\tau - T_x{}^\tau\}$$

and $dS = \sqrt{|\det \gamma|} dx \wedge d^{n-1}v$ is the surface element denoted in equations (5.28) and (5.23) by $dx\, d^{n-1}v_{t,x}$. Since $\eta_0 \sqrt{\det \mathfrak{g}} = \sqrt{\det \gamma}$ on $\mathbf{H}_{\lambda,\tau}$, we obtain that (remember that $\eta^\mu \partial_\mu$ is past directed)

$$L_1 = -E_\lambda[u(\tau)] \, . \tag{5.35}$$

From this, the sign coming from the Stokes' identity shows that

$$L_2 = E_\lambda[u(\tau_0)] \, . \tag{5.36}$$

On the hypersurfaces S_λ and S, since the unit outward normal is also past directed and the vector field $Y^\nu \partial_\nu = \partial_\tau - \partial_x$ future directed, we deduce from the positivity of the stress energy tensor that:

$$L_3 \leq 0 \quad \text{and} \quad L_4 \leq 0 \, . \tag{5.37}$$

We can now rewrite (5.34) as:

$$-E_\lambda[u(\tau)] + E_\lambda[u(\tau_0)] + L_3 + L_4 = \int_{\mathscr{U}_{\lambda,\tau}} \nabla_\mu \{ x^{-2\alpha-1} T_\nu{}^\mu Y^\nu \} dV \, . \tag{5.38}$$

Now, let us consider the right-hand side of the above equation. We have:

$$
\begin{aligned}
\nabla_\mu & \{ x^{-2\alpha-1} T_\nu{}^\mu Y^\nu \} \\
&= x^{-2\alpha-1} \Big\{ (\nabla_\mu T_\nu{}^\mu) Y^\nu + T_\nu{}^\mu (\nabla_\mu Y^\nu) - (2\alpha+1) x^{-1} T_\nu{}^\mu Y^\nu \nabla_\mu(x) \Big\} \\
&= x^{-2\alpha-1} \Big\{ (\nabla_\mu T_\nu{}^\mu) Y^\nu + T_\nu{}^\mu \{ \Gamma^\nu_{\mu\tau} - \Gamma^\nu_{\mu x} \} \\
&\quad - (2\alpha+1) x^{-1} \nabla_\mu x \, \{ T_\tau{}^\mu - T_x{}^\mu \} \Big\} \\
&=: R_1 + R_2 + R_3, \tag{5.39}
\end{aligned}
$$

where

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} \mathfrak{g}^{\sigma\rho} (\partial_\mu \mathfrak{g}_{\sigma\nu} + \partial_\nu \mathfrak{g}_{\mu\sigma} - \partial_\sigma \mathfrak{g}_{\mu\nu}),$$

are the Christoffel's symbols of the metric $\mathfrak{g}$. From (5.21), we have:

$$
\begin{aligned}
x^{2\alpha+1} |R_1| = |F| |\nabla_\nu u Y^\nu| = |F| |(\partial_\tau u - \partial_x u)| & \leq \frac{1}{2} \Big\{ F^2 + (\partial_\tau u - \partial_x u)^2 \Big\}) \\
& \leq c(\epsilon_0) \left(F^2 + |T_\tau{}^\tau - T_x{}^\tau| \right) \, .
\end{aligned}
\tag{5.40}
$$

As far as the second term is concerned, we have:

$$T_\nu{}^\mu \Gamma^\nu_{\mu\theta} = \frac{1}{2} T^{\mu\sigma} \partial_\theta \mathfrak{g}_{\mu\sigma} = -\frac{1}{2} T_{\mu\sigma} \partial_\theta \mathfrak{g}^{\mu\sigma}.$$

Thus, replacing successively in the above expression θ with τ and x and subtracting the two expressions we find that

$$x^{2\alpha+1}R_2 = -\frac{1}{2}T_\mu{}^\nu \mathfrak{g}_{\nu\sigma}\left(\partial_\tau - \partial_x\right)\mathfrak{g}^{\mu\sigma}.$$

From (5.26) we obtain:

$$x^{2\alpha+1}|R_2| = |T_{\nu\sigma}\left(\partial_\tau - \partial_x\right)\mathfrak{g}^{\mu\sigma}| \leq (n+1)C|\mathfrak{g}^\sharp|\left(|\mathfrak{g}|^2 + |(\partial_\tau - \partial_x)\mathfrak{g}^\sharp|^2\right)|T_\tau{}^\tau - T_x{}^\tau|.$$
(5.41)

For the third term we have, keeping in mind (5.6):

$$
\begin{aligned}
x^{2\alpha+1}R_3 \;&=\; -(2\alpha+1)x^{-1}T_\nu{}^\mu\nabla_\mu x\,Y^\nu \\
&=\; -(2\alpha+1)x^{-1}\mathfrak{g}^{\mu\sigma}T_{\nu\sigma}\nabla_\mu x\,Y^\nu \\
&=\; -(2\alpha+1)x^{-1}T_{\mu\nu}\nabla^\mu x\,Y^\nu \\
&=\; \underbrace{-(2\alpha+1)x^{-1}T_{\mu\nu}Y^\nu\omega^{(1)\mu}}_{\geq 0} - (2\alpha+1)\frac{\beta(x)}{x}T_{\mu\nu}Y^\nu\omega^{(2)\mu} \quad \text{for} \quad \alpha \leq -1/2 \\
&\geq\; -(2\alpha+1)\frac{\beta(x)}{x}T_{\mu\nu}Y^\nu\omega^{(2)\mu} = -(2\alpha+1)\frac{\beta(x)}{x}\left(T_{\mu\tau} - T_{\mu x}\right)\omega^{(2)\mu} \\
&\geq\; -C(\,\alpha, C_0, n)|\mathfrak{h}^\sharp|\left(1 + |\mathfrak{g}|^2 + |\mathfrak{g}^\sharp|^2\right)|T_\tau{}^\tau - T_x{}^\tau|.
\end{aligned}
$$
(5.42)

Let us justify the last inequality. In other words let us show that the expression $T_{\mu\tau} - T_{\mu x}$ is controlled by $|T_\tau{}^\tau - T_x{}^\tau|$. We have:

$$
\begin{aligned}
|T_{\mu\tau} - T_{\mu x}| \;&=\; |\partial_\mu u\left(\partial_\tau - \partial_x\right)u - \frac{1}{2}\left(\mathfrak{g}_{\mu\tau} - \mathfrak{g}_{\mu x}\right)\mathfrak{g}^{\alpha\beta}\partial_\alpha u\partial_\beta u| \\
&\leq\; (\partial_\mu u)^2 + [(\partial_\tau - \partial_x)\,u]^2 + \left(|\mathfrak{g}|^2 + |\mathfrak{g}^\sharp|^2\right)\left(\delta^{\alpha\beta}\partial_\alpha u\partial_\beta u\right) \\
&\leq\; C(\epsilon_0)\left(1 + |\mathfrak{g}|^2 + |\mathfrak{g}^\sharp|^2\right)|T_\tau{}^\tau - T_x{}^\tau|. \qquad \text{See (5.24)}
\end{aligned}
$$

Inequalities (5.40), (5.41) and (5.42) show that the right-hand side of (5.39) can be estimated as:

$$R_1+R_2+R_3 \geq -C_1 x^{-(2\alpha+1)}\left\{\left(1 + |\mathfrak{h}^\sharp| + |\mathfrak{g}^\sharp|\right)\left(1 + |\mathfrak{g}|^2 + |\mathfrak{g}^\sharp|^2 + |(\partial_\tau - \partial_x)\mathfrak{g}^\sharp|^2\right)|T^Y| + F^2\right\},$$
(5.43)

where $C_1 = C(\alpha, \epsilon_0, C_0, n)$. Now from (5.38) we have

$$-E_\lambda[u(t)] + E_\lambda[u(\tau_0)] + L_3 + L_4 = R_1 + R_2 + R_3,$$

thus, using (5.37), we obtain the following:

$$E_\lambda[u(t)] \leq E_\lambda[u(\tau_0)] \quad + \quad C_1 \int_{\tau_0}^{t} \int_{\mathbf{H}_{\lambda,s}} x^{-2\alpha} \left\{ \left(1 + |\mathfrak{h}^\sharp| + |\mathfrak{g}^\sharp|\right) \left(1 + |\mathfrak{g}|^2 + |\mathfrak{g}^\sharp|^2 \right. \right.$$

$$+ \quad \left. \left. |(\partial_\tau - \partial_x)\mathfrak{g}^\sharp|^2\right) |T^Y| + F^2(s)\right\} ds \frac{dx}{x} d^{n-1}\nu .$$

Therefore, there exists a constant $C_1 > 0$ depending upon n, ϵ_0, α and C_0 such that

$$E_\lambda[u(\tau)] \leq C_1 \left\{ E_\lambda[u(\tau_0)] + \int_{\tau_0}^{\tau} \left\{ \|F(s)\|_{\mathscr{H}_0^\alpha(\mathbf{H}_{\lambda,s})}^2 + \left(1 + \|\mathfrak{h}^\sharp\|_{L^\infty} + \|\mathfrak{g}^\sharp\|_{L^\infty}\right) \right.\right.$$

$$\left.\left. \times \left(1 + \|\mathfrak{g}\|_{L^\infty(\mathbf{H}_{\lambda,s})}^2 + \|\mathfrak{g}^\sharp\|_{L^\infty(\mathbf{H}_{\lambda,s})}^2 + \| (\partial_\tau - \partial_x)\, \mathfrak{g}^\sharp\|_{L^\infty(\mathbf{H}_{\lambda,\tau})}^2 \right) E_\lambda[u(s)]\right\} ds \right\}$$

$$(5.44)$$

and the proof is completed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

5.2.3 Estimates on the higher space derivatives of the solution

To proceed further, we would like to have an estimate similar to (5.30) on space derivatives of the unknown function in equation (5.1). For this purpose, for $k \in \mathbb{N}$, $\beta = (\beta_1, \beta_2, \ldots, \beta_r) \in \mathbb{N}^r$, with $|\beta| \leq k$; we set:

$$\overset{(\beta)}{T}{}_\nu{}^\mu = x^{-2\alpha - 1 + 2\beta_1} \left\{ \nabla^\mu \mathscr{D}^\beta u \nabla_\nu \mathscr{D}^\beta u - \frac{1}{2}\delta_\nu{}^\mu \nabla^\alpha \mathscr{D}^\beta u \nabla_\alpha \mathscr{D}^\beta u \right\},$$

where $\alpha \leq -1/2$ is the real parameter of the previous section, $\mathscr{D}^\beta = X_1^{\beta_1} X_2^{\beta_2} \ldots X_r^{\beta_r}$, with the X_i's being the vector fields defined in [13, page 51]: for $i = 2, \ldots, r$, $X_i = \sum\limits_{A=2}^{r} X_i^A(v)\partial_A$, where the X_i^A's are smooth functions bounded on bounded set with all their derivatives, and $X_1 = \partial_x$. Since the operator ∇ is linear, as in (5.21), we have

$$\nabla_\mu \overset{(\beta)}{T}{}_\nu{}^\mu = x^{2\alpha - 1 + 2\beta_1} \Box_\mathfrak{g}(\mathscr{D}^\beta u)\nabla_\nu(\mathscr{D}^\beta u) + (-2\alpha - 1 + 2\beta_1)\frac{\nabla_\mu(x)}{x}\overset{(\beta)}{T}{}_\nu{}^\mu .$$

Now

$$\Box_\mathfrak{g}(\mathscr{D}^\beta u) = \mathscr{D}^\beta(\Box_\mathfrak{g} u) + [\Box_\mathfrak{g}, \mathscr{D}^\beta]u = \mathscr{D}^\beta F + [\Box_\mathfrak{g}, \mathscr{D}^\beta]u , \qquad (5.45)$$

for any solution of the equation (5.1). Thus

$$\nabla_\mu \overset{(\beta)}{T}_\nu{}^\mu = x^{-2\alpha-1+2\beta_1} \left\{ \mathscr{D}^\beta F + [\Box_{\mathfrak{g}}, \mathscr{D}^\beta] u \right\} \nabla_\nu (\mathscr{D}^\beta u) + (-2\alpha-1+2\beta_1) \frac{\nabla_\mu(x)}{x} \overset{(\beta)}{T}_\nu{}^\mu .$$
$$(5.46)$$

Similarly to the previous section, we set:

$$\overset{(\beta)}{T}{}^Y = \overset{(\beta)}{T}_\tau{}^\tau - \overset{(\beta)}{T}_x{}^\tau ,$$

$$E_k^\alpha[u(\tau)] = \sum_{|\beta|=0}^{k} \int_{\mathbf{H}_t} -\overset{(\beta)}{T}{}^Y \, dx \, d^{n-1}\nu_{t,x} \quad \text{and} \quad E_{k,\lambda}^\alpha[u(t)] = \sum_{|\beta|=0}^{k} \int_{\mathbf{H}_{\lambda,\tau}} -\overset{(\beta)}{T}{}^Y \, dx \, d^{n-1}\nu_{t,x} .$$
$$(5.47)$$

REMARK 5.6 From (5.24) we deduce the following decomposition for $\overset{(\beta)}{T}{}^Y$:

$$\overset{(\beta)}{T}{}^Y = -\frac{1}{2} \left\{ -\mathfrak{g}^{\tau\tau} \left(x^{-\alpha-\frac{1}{2}+\beta_1} \mathscr{D}^\beta (\partial_\tau - \partial_x) u \right)^2 \right.$$

$$+ \lambda \left(x^{-\alpha-\frac{1}{2}+\beta_1} \mathscr{D}^\beta(\partial_x u) + \frac{(\mathfrak{g}^{xA} + \mathfrak{g}^{A\tau})}{\lambda} \partial_A \left(x^{-\alpha-\frac{1}{2}+\beta_1} \mathscr{D}^\beta u \right) \right)^2$$

$$\left. + \left(\mathfrak{g}^{AB} - \kappa^{AB} \right) \partial_A \left(x^{-\alpha-\frac{1}{2}+\beta_1} \mathscr{D}^\beta u \right) \partial_B \left(x^{-\alpha-\frac{1}{2}+\beta_1} \mathscr{D}^\beta u \right) \right\} . \quad (5.48)$$

Since the coefficients of the terms arising in commuting ∂_A and $\mathscr{D}^\beta$ are uniformly bounded, from the above we find that the energy of order k controls the $\mathscr{H}_k^\alpha$-norms of the first order derivatives of the unknown function u. That is:

$$\|(\partial_\tau - \partial_x)u\|_{\mathscr{H}_k^\alpha(\mathbf{H}_{\lambda,\tau})}^2 + \|\partial_x u\|_{\mathscr{H}_k^\alpha(\mathbf{H}_{\lambda,\tau})}^2 + \sum_A \|\partial_A u\|_{\mathscr{H}_k^\alpha(\mathbf{H}_{\lambda,\tau})}^2 \le C E_{k,\lambda}^\alpha[u(\tau)] .$$
$$(5.49)$$

Let us set

$$\Upsilon^\nu := -\mathfrak{g}^{\alpha\mu} \Gamma^\nu_{\alpha\mu} = \frac{1}{\sqrt{|\det \mathfrak{g}|}} \partial_\mu \left(\sqrt{|\det \mathfrak{g}|} \mathfrak{g}^{\mu\nu} \right) . \quad (5.50)$$

Let us define

$$M(\tau) := \|F\|_{\mathscr{B}_0^\alpha(\mathbf{H}_{\lambda,\tau})}^2 + \|(\mathfrak{g}, (\partial_\tau - \partial_x)\mathfrak{g}^\sharp)\|_{L^\infty(\mathbf{H}_{\lambda,\tau})}^2$$
$$+ \|(\mathfrak{g}^\sharp, \mathfrak{h}^\sharp, \Upsilon)\|_{\mathscr{C}_{\{x=0\},1}^0(\mathbf{H}_{\lambda,\tau})}^2 . \quad (5.51)$$

We claim that:

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PROPOSITION 5.7 *Let $\lambda > 0$, $k \in \mathbb{N}$ and suppose that $\alpha \leq -\frac{1}{2}$. Under hypotheses (5.2)-(5.4) and (5.25), there exists a function $C_2(\epsilon_0, C_0, \alpha, k, n, M)$. monotonously increasing in M, which we write as $C_2(M)$, such that for all*

$$\tau \in [\tau_0, \tau_1]$$

and for all u satisfying (5.1) we have

$$E_{k,\lambda}^{\alpha}[u(\tau)]$$

$$\leq E_{k,\lambda}^{\alpha}[u(\tau_0)] + \int_{\tau_0}^{\tau} C_2(M(s)) \Bigg\{ E_{k,\lambda}^{\alpha}[u(s)] + \|F(s)\|_{\mathscr{H}_k^{\alpha}(\mathbf{H}_{\lambda,\tau})}^2$$

$$+ \|\big((\partial_\tau - \partial_x)u, \partial_x u, \partial_A u\big)\|_{\mathscr{B}_1^{\alpha}(\mathbf{H}_{\lambda,\tau})}^2 \times \|\big(\mathfrak{g}^{\sharp}, \mathfrak{h}^{\sharp}, \Upsilon\big)\|_{\mathscr{G}_k^0(\mathbf{H}_{\lambda,\tau})}^2 \Bigg\} ds .$$

$$(5.52)$$

REMARK 5.8 *The reader should note that C_2 does not depend upon λ.*

PROOF: If the right-hand side of (5.52) is infinite there is nothing to prove. Otherwise, the calculations that follow should be done assuming smoothness of u, and the inequality for general u's can be obtained by a density argument.

The equivalent of (5.34) for space-derivatives of the solution of (5.1) reads:

$$\sum_{|\beta|=0}^{k} \int_{\mathbf{H}_{\lambda,\tau}} \overset{(\beta)}{T_\nu}{}^\mu Y^\nu \eta_\mu dS \quad + \quad \sum_{|\beta|=0}^{k} \int_{\mathbf{H}_{\lambda,\tau_0}} \overset{(\beta)}{T_\nu}{}^\mu Y^\nu \eta_\mu dS + \sum_{|\beta|=0}^{k} \int_{S_{\lambda,\tau}} \overset{(\beta)}{T_\nu}{}^\mu Y^\nu \eta_\mu dS$$

$$+ \quad \sum_{|\beta|=0}^{k} \int_{S_{\theta,\tau}} \overset{(\beta)}{T_\nu}{}^\mu Y^\nu \eta_\mu dS = \sum_{|\beta|=0}^{k} \int_{\mathscr{U}_{\lambda,\tau}} \nabla_\mu \left(\overset{(\beta)}{T_\nu}{}^\mu Y^\nu \right) dV \quad (5.53)$$

which gives the following equation:

$$-E_{k,\lambda}^{\alpha}[u(\tau)] + E_{k,\lambda}^{\alpha}[u(\tau_0)] \quad + \quad \underbrace{\sum_{|\beta|=0}^{k} \int_{S_{\lambda,\tau}} \overset{(\beta)}{T_\nu}{}^\mu Y^\nu \eta_\mu dS + \sum_{|\beta|=0}^{k} \int_{S_{\theta,\tau}} \overset{(\beta)}{T_\nu}{}^\mu Y^\nu \eta_\mu dS}_{:= \hat{L}_3 + \hat{L}_4 \leq 0}$$

$$= \sum_{|\beta|=0}^{k} \int_{\mathscr{U}_{\lambda,\tau}} \nabla_\mu \{ \overset{(\beta)}{T_\nu}{}^\mu Y^\nu \} dx \, d\nu. \quad (5.54)$$

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Again as in the previous section we take $Y^\nu \partial_\nu = \partial_\tau - \partial_x$, then the divergence in the right-hand side of (5.54) reads:

$$
\begin{aligned}
\nabla_\mu \{ \overset{(\beta)}{T_\nu}{}^\mu Y^\nu \}
&= \nabla_\mu \overset{(\beta)}{T_\nu}{}^\mu Y^\nu + \overset{(\beta)}{T_\nu}{}^\mu \nabla_\mu Y^\nu \\
&= x^{-2\alpha - 1 - 2\beta_1} \left\{ \mathscr{D}^\beta F + [\Box_{\mathfrak{g}}, \mathscr{D}^\beta] u \right\} (\partial_\tau - \partial_x) (\mathscr{D}^\beta u) \\
&\quad + \overset{(\beta)}{T_\nu}{}^\mu \left(\Gamma^\nu_{\mu\tau} - \Gamma^\nu_{\mu x} \right) + (-2\alpha - 1 + 2\beta_1) \frac{\nabla_\mu(x)}{x} \left(\overset{(\beta)}{T_\tau}{}^\mu - \overset{(\beta)}{T_x}{}^\mu \right) \\
&=: \ \widehat{R}_1 + \widehat{R}_2 + \widehat{R}_3.
\end{aligned}
\tag{5.55}
$$

If we repeat the calculations in the previous section that led to (5.41) and (5.42), we obtain that there exists a constant $C = C(n, k, C_0, \alpha, \epsilon_0) > 0$ such that:

$$
|\widehat{R}_2| \le C |\mathfrak{g}^\sharp| \left(|\mathfrak{g}|^2 + |(\partial_\tau - \partial_x)\mathfrak{g}^\sharp|^2 \right) |\overset{(\beta)}{T}{}^Y|
\tag{5.56}
$$

and, keeping in mind that the term with the worst power of x can be discarded because of a favorable sign,

$$
\hat{R}_3 \ge -C |\mathfrak{h}^\sharp| \left(1 + |\mathfrak{g}|^2 + |\mathfrak{g}^\sharp|^2 \right) |\overset{(\beta)}{T}{}^Y| .
\tag{5.57}
$$

As far as the term $\widehat{R}_1$ is concerned, from the inequality $ab \le \frac{1}{2}(a^2 + b^2)$, we have:

$$
\begin{aligned}
x^{2\alpha + 1 - 2\beta_1} |\widehat{R}_1|
&= |\{ \mathscr{D}^\beta F + [\Box_{\mathfrak{g}}, \mathscr{D}^\beta] u \} (\partial_\tau - \partial_x)(\mathscr{D}^\beta u)| \\
&\le \frac{1}{2} (\mathscr{D}^\beta F)^2 + \frac{1}{2} \left(\left[\Box_{\mathfrak{g}}, \mathscr{D}^\beta \right] u \right)^2 + \left[(\partial_\tau - \partial_x)\left(\mathscr{D}^\beta u \right) \right]^2 \\
&\le (\mathscr{D}^\beta F)^2 + C(\epsilon_0) |\overset{(\beta)}{T}{}^Y| + \left(\left[\Box_{\mathfrak{g}}, \mathscr{D}^\beta \right] u \right)^2 .
\end{aligned}
\tag{5.58}
$$

From inequalities (5.56)–(5.58) and the fact that $\hat{L}_3 \le 0$ and $\hat{L}_4 \le 0$ we obtain that:

$$
\begin{aligned}
E^\alpha_{k,\lambda}[u(\tau)] - E^\alpha_{k,\lambda}[u(\tau_0)]
&\le C \int_{\tau_0}^\tau \left[\left(1 + \|\mathfrak{h}^\sharp\|_{L^\infty} + \|\mathfrak{g}^\sharp\|_{L^\infty} \right) \left(1 + \|\mathfrak{g}\|^2_{L^\infty} + \|\mathfrak{g}^\sharp\|^2_{L^\infty} \right. \right. \\
&\quad \left. \left. + \|(\partial_\tau - \partial_x)\mathfrak{g}^\sharp\|^2_{L^\infty} \right) E^\alpha_{k,\lambda}[u(s)] + \|F(s)\|^2_{\mathscr{H}^\alpha_k(\mathbf{H}_{\lambda,s})} \right] ds \\
&\quad + \sum_{|\beta|=0}^k \int_{\tau_0}^\tau \int_{\mathbf{H}_{\lambda,s}} x^{-2\alpha - 1 + 2\beta_1} ([\Box_{\mathfrak{g}}, \mathscr{D}^\beta] u)^2(s) dx \, d\nu_{t,x} ds
\end{aligned}
\tag{5.59}
$$

with $C = C(n, \alpha, k, C_0, \epsilon_0)$. Now, let us estimate the last term of the right-hand side of the above inequality. From the definition (5.50) of Υ^ν we have

$$\Box_{\mathfrak{g}} = \mathfrak{g}^{\mu\nu} \partial^2_{\mu\nu} + \Upsilon^\nu \partial_\nu, \qquad (5.60)$$

and then

$$
\begin{aligned}
[\Box_{\mathfrak{g}}, \mathscr{D}^\beta]u &= \mathfrak{g}^{\alpha\mu}[\partial_\alpha\partial_\mu, \mathscr{D}^\beta]u - \Upsilon^\nu[\mathscr{D}^\beta, \partial_\nu]u - \left\{ \mathscr{D}^\beta\left(\Upsilon^\nu\partial_\nu u\right) - \Upsilon^\nu\mathscr{D}^\beta\left(\partial_\nu u\right) \right\} \\
&\quad - \left\{ \mathscr{D}^\beta\left(\mathfrak{g}^{\alpha\mu}\partial_\alpha\partial_\mu u\right) - \mathfrak{g}^{\alpha\mu}\mathscr{D}^\beta\left(\partial_\alpha\partial_\mu u\right) \right\} \\
&=: A_1 + A_2 + A_3 + A_4 .
\end{aligned}
\qquad (5.61)
$$

To estimate the first and second terms, we use the explicit form of the differential operator $\mathscr{D}$: $\mathscr{D}^\beta = \partial_x^{\beta_1} X_2^{\beta_2} \dots X_r^{\beta_r} = \partial_x^{\beta_1} X_v^{\beta_v}$. Since ∂_τ and ∂_x commute with $\mathscr{D}^\beta$, we have (see (5.27))

$$A_1 = \mathfrak{g}^{\mu\alpha}[\partial_\mu\partial_\alpha, \mathscr{D}^\beta]u = 2\mathfrak{g}^{\tau A}[(\partial_\tau - \partial_x)\partial_A, \mathscr{D}^\beta]u + 2(\mathfrak{g}^{xA} + \mathfrak{g}^{\tau A})[\partial_x\partial_A, \mathscr{D}^\beta]u + \mathfrak{g}^{AB}[\partial_A\partial_B, \mathscr{D}^\beta]u,$$

and since

$$\mathfrak{g}^{\tau A}[(\partial_\tau - \partial_x)\partial_A, \mathscr{D}^\beta]u = \mathfrak{g}^{\tau A}\partial_x^{\beta_1}\partial_A X_v^{\beta_v}[(\partial_\tau - \partial_x)u] - \mathfrak{g}^{\tau A}\partial_x^{\beta_1} X_v^{\beta_v}\partial_A[(\partial_\tau - \partial_x)u]$$

we obtain that (see (5.49):

$$
\begin{aligned}
\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1}\left(\mathfrak{g}^{\tau A}[(\partial_\tau - \partial_x)\partial_A, \mathscr{D}^\beta]u\right)^2 dx\, d\nu &\leq c\|\mathfrak{g}^\sharp\|^2_{L^\infty(\mathbf{H}_{\lambda,\tau})}\|(\partial_\tau - \partial_x)u\|^2_{\mathscr{H}^\alpha_k} \\
&\leq c\|\mathfrak{g}^\sharp\|^2_{L^\infty(\mathbf{H}_{\lambda,\tau})}E^\alpha_{k,\lambda}[u(\tau)] .
\end{aligned}
$$

Similarly, we have

$$\left(\mathfrak{g}^{xA} + \mathfrak{g}^{\tau A}\right)[\partial_x\partial_A, \mathscr{D}^\beta]u = \left(\mathfrak{g}^{xA} + \mathfrak{g}^{\tau A}\right)\left(\partial_A\mathscr{D}^\beta(\partial_x u) - \mathscr{D}^\beta\partial_A(\partial_x u)\right) ,$$

which leads to:

$$
\begin{aligned}
\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1}\left\{\left(\mathfrak{g}^{xA} + \mathfrak{g}^{\tau A}\right)[\partial_x\partial_A, \mathscr{D}^\beta]u\right\}^2(s)dx\, d\nu &\leq C\|\mathfrak{g}^\sharp\|^2_{L^\infty(\mathbf{H}_{\lambda,\tau})}\|\partial_x u\|^2_{\mathscr{H}^\alpha_k} \\
&\leq c\|\mathfrak{g}^\sharp\|^2_{L^\infty(\mathbf{H}_{\lambda,\tau})}E^\alpha_{k,\lambda}[u(\tau)] .
\end{aligned}
$$

Similar calculations give:

$$
\begin{aligned}
\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1}(\mathfrak{g}^{AB}[\partial_A\partial_B, \mathscr{D}^\beta]u)^2(s)dx\, d\nu &\leq c\|\mathfrak{g}^\sharp\|^2_{L^\infty(\mathbf{H}_{\lambda,\tau})}\sum_A \|\partial_A u\|^2_{\mathscr{H}^\alpha_k} \\
&\leq c\|\mathfrak{g}^\sharp\|^2_{L^\infty(\mathbf{H}_{\lambda,\tau})}E^\alpha_{k,\lambda}[u(\tau)] .
\end{aligned}
$$

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We obtain thus the following estimate for the first term of the identity (5.61):

$$\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1} A_1^2 dx\, d\nu \leq C\|\mathfrak{g}^\sharp\|_{L^\infty(\mathbf{H}_{\lambda,\tau})}^2 E_{k,\lambda}^\alpha[u(\tau)] . \qquad (5.62)$$

Again since ∂_τ and ∂_x commute with $\mathscr{D}^\beta$, if we develop the second term of (5.61), we find that:

$$A_2 = \Upsilon^\nu[\mathscr{D}^\beta, \partial_\nu]u = \Upsilon^A[\mathscr{D}^\beta, \partial_A]u$$

and we then have the estimates:

$$\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1} A_2^2 dx\, d\nu \leq \|\Upsilon^A\|_{L^\infty}^2 \|\partial_A u\|_{\mathscr{H}_{k-1}^\alpha}^2 \leq \|\Upsilon^A\|_{L^\infty}^2 E_{k,\lambda}^\alpha[u(\tau)] . \qquad (5.63)$$

As far as the third term is concerned, we write

$$
\begin{aligned}
A_3 = \mathscr{D}^\beta\left(\Upsilon^\nu\partial_\nu u\right) - \Upsilon^\nu\mathscr{D}^\beta\left(\partial_\nu u\right) \quad &= \quad \mathscr{D}^\beta\left(\Upsilon^\tau(\partial_\tau - \partial_x)u\right) - \Upsilon^\tau\mathscr{D}^\beta\left((\partial_\tau - \partial_x)u\right) \\
&\quad + \mathscr{D}^\beta\left((\Upsilon^x + \Upsilon^\tau)\partial_x u\right) - (\Upsilon^x + \Upsilon^\tau)\mathscr{D}^\beta\left(\partial_x u\right) \\
&\quad + \mathscr{D}^\beta\left(\Upsilon^A\partial_A u\right) - \Upsilon^A\mathscr{D}^\beta\left(\partial_A u\right) \\
&=: \quad I + II + III .
\end{aligned}
$$

Now we will use the weighted Moser-type inequality (A.35) of Proposition A.3 of [13] to estimate the terms of A_3. Its first term gives the following

$$
\begin{aligned}
&\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1}\left\{I\right\}^2 dx\, d\nu \\
&= \left\|x^{\beta_1}\mathscr{D}^\beta\left(\Upsilon^\tau(\partial_\tau - \partial_x)u\right) - x^{\beta_1}\Upsilon^\tau\mathscr{D}^\beta\left((\partial_\tau - \partial_x)u\right)\right\|_{\mathscr{H}_0^\alpha(\mathbf{H}_{\lambda,\tau})}^2 \\
&\leq C_s\left(\|(\partial_\tau - \partial_x)u\|_{\mathscr{B}_0^\alpha}^2\|\Upsilon^\tau\|_{\mathscr{G}_k^0}^2 + \|(\partial_\tau - \partial_x)u\|_{\mathscr{H}_{k-1}^\alpha}^2\|\Upsilon^\tau\|_{\mathscr{C}_{\{x=0\},1}^0}^2\right) \\
&\leq C\left(\|(\partial_\tau - \partial_x)u\|_{\mathscr{B}_0^\alpha}^2\|\Upsilon^\tau\|_{\mathscr{G}_k^0}^2 + \|\Upsilon^\tau\|_{\mathscr{C}_{\{x=0\},1}^0}^2 E_k^\alpha[u(\tau)]\right) . \qquad (5.64)
\end{aligned}
$$

For the second term:

$$
\begin{aligned}
&\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1}\left\{II\right\}^2 dx\, d\nu \\
&= \left\|x^{\beta_1}\mathscr{D}^\beta\left(\Upsilon^x + \Upsilon^\tau\right)\partial_x u - x^{\beta_1}(\Upsilon^x + \Upsilon^\tau)\mathscr{D}^\beta(\partial_x u)\right\|_{\mathscr{H}_0^\alpha(\mathbf{H}_{\lambda,\tau})}^2 \\
&\leq C_s\left(\|\partial_x u\|_{\mathscr{B}_0^\alpha}^2\|\Upsilon^x + \Upsilon^\tau\|_{\mathscr{G}_k^0}^2 + \|\partial_x u\|_{\mathscr{H}_{k-1}^\alpha}^2\|\Upsilon^x + \Upsilon^\tau\|_{\mathscr{C}_{\{x=0\},1}^0}^2\right) \\
&\leq C\left(\|\partial_x u\|_{\mathscr{B}_0^\alpha}^2\|\Upsilon^x + \Upsilon^\tau\|_{\mathscr{G}_k^0}^2 + \|\Upsilon^x + \Upsilon^\tau\|_{\mathscr{C}_{\{x=0\},1}^0}^2 E_k^\alpha[u(\tau)]\right) .
\end{aligned}
$$

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The same holds for the third term of A_3:

$$\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1} \{III\}^2 \, dx \, d\nu = \|x^{\beta_1} \mathscr{D}^\beta (\Upsilon^A \partial_A u) - x^{\beta_1} \Upsilon^A \mathscr{D}^\beta (\partial_A u)\|^2_{\mathscr{H}_0^\alpha(\mathbf{H}_{\lambda,\tau})}$$

$$\leq C_s \left(\|\partial_A u\|^2_{\mathscr{B}_0^\alpha} \|\Upsilon^A\|^2_{\mathscr{G}_k^0} + \|\partial_A u\|^2_{\mathscr{H}_{k-1}^\alpha} \|\Upsilon^A\|^2_{\mathscr{C}_{\{x=0\},1}^0} \right)$$

$$\leq C \left(\|\partial_A u\|^2_{\mathscr{B}_0^\alpha} \|\Upsilon^A\|^2_{\mathscr{G}_k^0} + \|\Upsilon^A\|^2_{\mathscr{C}_{\{x=0\},1}^0} E_k^\alpha[u(\tau)] \right).$$

We then obtain the following estimate for the third term of equation (5.61)

$$\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1} (A_3)^2 dx \, d\nu$$

$$\leq \|(\partial_\tau - \partial_x) u\|^2_{\mathscr{B}_0^\alpha} \|\Upsilon^\tau\|^2_{\mathscr{G}_k^0} + \|\partial_x u\|^2_{\mathscr{B}_0^\alpha} \|\Upsilon^\tau + \Upsilon^x\|^2_{\mathscr{G}_k^0} + \|\partial_A u\|^2_{\mathscr{B}_0^\alpha} \|\Upsilon^A\|^2_{\mathscr{G}_k^0}$$

$$+ \left\{ \|\Upsilon^\tau\|^2_{\mathscr{C}_{\{x=0\},1}^0} + \|\Upsilon^x + \Upsilon^\tau\|^2_{\mathscr{C}_{\{x=0\},1}^0} + \|\Upsilon^A\|^2_{\mathscr{C}_{\{x=0\},1}^0} \right\} E_{k,\lambda}^\alpha[u(\tau)].$$

$$(5.65)$$

In order to estimate the fourth term A_4 of (5.61), we need to look separately at each of its components as we have to make sure that every ∂_x^2 comes with a factor of x. We write

$$A_4 = A^{00} + 2A^{\tau x} + 2A^{\tau A} + A^{xx} + 2A^{xA} + A^{AB}, \qquad (5.66)$$

where the labeling A^{ab} corresponds to the terms obtained when in A_4 we replace $\mathfrak{g}^{\alpha\beta} \partial_{\alpha\beta}^2$ with its expression as in (5.27). Now we use again the weighted Moser type inequality of Proposition A.3 of [13] to estimate these terms. We have:

$$\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1} \{A^{AB}\}^2 dx \, d\nu$$

$$= \|x^{\beta_1} \mathscr{D}^\beta (\mathfrak{g}^{AB} \partial_A \partial_B u) - x^{\beta_1} \mathfrak{g}^{AB} \mathscr{D}^\beta (\partial_A \partial_B u)\|_{\mathscr{H}_0^\alpha(\mathbf{H}_{\lambda,\tau})}$$

$$\leq C_s \sum_A \left(\|\partial_A u\|^2_{\mathscr{B}_1^\alpha} \|\mathfrak{g}^\sharp\|^2_{\mathscr{G}_k^0} + \|\partial_A u\|^2_{\mathscr{H}_k^\alpha} \|\mathfrak{g}^\sharp\|^2_{\mathscr{C}_{\{x=0\},1}^0} \right)$$

$$\leq C \left(\sum_A \|\partial_A u\|^2_{\mathscr{B}_1^\alpha} \|\mathfrak{g}^\sharp\|^2_{\mathscr{G}_k^0} + \|\mathfrak{g}^\sharp\|^2_{\mathscr{C}_{\{x=0\},1}^0} E_{k,\lambda}^\alpha[u(\tau)] \right), \qquad (5.67)$$

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and

$$\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1} \left\{ A^{\tau A} \right\}^2 dx\, d\nu$$

$$= \| x^{\beta_1} \mathscr{D}^\beta \left(\mathfrak{g}^{\tau A} \partial_A (\partial_\tau - \partial_x) u \right) - x^{\beta_1} \mathfrak{g}^{\tau A} \mathscr{D}^\beta \left(\partial_A (\partial_\tau - \partial_x) u \right) \|_{\mathscr{H}_0^\alpha}$$

$$\leq C_s \left(\| (\partial_\tau - \partial_x) u \|_{\mathscr{B}_1^\alpha}^2 \| \mathfrak{g}^\sharp \|_{\mathscr{G}_k^0}^2 + \| (\partial_\tau - \partial_x) u \|_{\mathscr{H}_k^\alpha}^2 \| \mathfrak{g}^\sharp \|_{\mathscr{C}_{\{x=0\},1}^0}^2 \right)$$

$$\leq C_s \left(\| (\partial_\tau - \partial_x) u \|_{\mathscr{B}_1^\alpha}^2 \| \mathfrak{g}^\sharp \|_{\mathscr{G}_k^0}^2 + \| \mathfrak{g}^\sharp \|_{\mathscr{C}_{\{x=0\},1}^0}^2 E_{k,\lambda}^\alpha[u(\tau)] \right). \qquad (5.68)$$

Continuing in this way we have:

$$\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1} \left\{ A^{xA} \right\}^2 dx\, d\nu$$

$$\leq \| x^{\beta_1} \mathscr{D}^\beta \left\{ \left(\mathfrak{g}^{xA} + \mathfrak{g}^{\tau A} \right) \partial_A \partial_x u \right\} - x^{\beta_1} \left(\mathfrak{g}^{xA} + \mathfrak{g}^{\tau A} \right) \mathscr{D}^\beta \partial_A \partial_x u \|_{\mathscr{H}_0^\alpha}$$

$$\leq C \left(\| \partial_A \partial_x u \|_{\mathscr{B}_0^\alpha}^2 \| \left(\mathfrak{g}^{xA} + \mathfrak{g}^{\tau A} \right) \|_{\mathscr{G}_k^0}^2 + \| \partial_A \partial_x u \|_{\mathscr{H}_{k-1}^\alpha}^2 \| \left(\mathfrak{g}^{xA} + \mathfrak{g}^{\tau A} \right) \|_{\mathscr{C}_{\{x=0\},1}^0}^2 \right)$$

$$\leq C \sum_A \left(\| \partial_x u \|_{\mathscr{B}_1^\alpha}^2 \| \left(\mathfrak{g}^{xA} + \mathfrak{g}^{\tau A} \right) \|_{\mathscr{G}_k^0}^2 + \| \partial_x u \|_{\mathscr{H}_k^\alpha}^2 \| \left(\mathfrak{g}^{xA} + \mathfrak{g}^{\tau A} \right) \|_{\mathscr{C}_{\{x=0\},1}^0}^2 \right)$$

$$\leq C \sum_A \left(\| \partial_x u \|_{\mathscr{B}_1^\alpha}^2 \| \left(\mathfrak{g}^{xA} + \mathfrak{g}^{\tau A} \right) \|_{\mathscr{G}_k^0}^2 + \| \left(\mathfrak{g}^{xA} + \mathfrak{g}^{\tau A} \right) \|_{\mathscr{C}_{\{x=0\},1}^0}^2 E_{k,\lambda}^\alpha[u(\tau)] \right). \quad (5.69)$$

We recall that $\mathfrak{g}^{\tau\tau} + \mathfrak{g}^{x\tau} = x\mathfrak{h}^1(\tau, x, v^A)$, we then obtain the following expression for $A^{\tau x}$.

$$A^{\tau x} = \mathscr{D}^\beta \left[\mathfrak{h}^1 x \partial_x (\partial_\tau - \partial_x) u \right] - x\mathfrak{h}^1 \mathscr{D}^\beta \left[\partial_x (\partial_\tau - \partial_x) u \right]$$

$$= \mathscr{D}^\beta \left[\mathfrak{h}^1 x \partial_x (\partial_\tau - \partial_x) u \right] - \mathfrak{h}^1 \mathscr{D}^\beta \left[x \partial_x (\partial_\tau - \partial_x) u \right]$$

$$+ \underbrace{ \mathfrak{h}^1 \mathscr{D}^\beta \left[x \partial_x (\partial_\tau - \partial_x) u \right] - x\mathfrak{h}^1 \mathscr{D}^\beta \left[\partial_x (\partial_\tau - \partial_x) u \right] }_{= \beta_1 \mathfrak{h}^1 \mathscr{D}^\beta (\partial_\tau - \partial_x) u}.$$

Since

$$\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1} \left\{ \mathfrak{h}^1 \mathscr{D}^\beta (\partial_\tau - \partial_x) u \right\}^2 dx\, d\nu \leq \| \mathfrak{h}^1 \|_{L^\infty}^2 \| (\partial_\tau - \partial_x) u \|_{\mathscr{H}_k^\alpha}^2 ,$$

we have

$$\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1} \left\{A^{\tau x}\right\}^2 dx\, d\nu$$

$$\leq\ C_s \left(\|(\partial_\tau - \partial_x)u)\|^2_{\mathscr{B}_1^\alpha} \|\mathfrak{h}^1\|^2_{\mathscr{G}_k^0} + \|(\partial_\tau - \partial_x)u\|^2_{\mathscr{H}_k^\alpha} \|\mathfrak{h}^1\|^2_{\mathscr{C}^0_{\{x=0\},1}} \right)$$

$$\leq\ C \left(\|(\partial_\tau - \partial_x)u)\|^2_{\mathscr{B}_1^\alpha} \|\mathfrak{h}^1\|^2_{\mathscr{G}_k^0} + \|\mathfrak{h}^1\|^2_{\mathscr{C}^0_{\{x=0\},1}} E^\alpha_{k,\lambda}[u(\tau)] \right). \qquad (5.70)$$

On the other hand, since $\mathfrak{g}^{\tau\tau} + 2\mathfrak{g}^{\tau x} + \mathfrak{g}^{xx} = 1 + x\mathfrak{h}$, we have

$$\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1} \left\{A^{xx}\right\}^2 dx\, d\nu\ \leq\ \|x^{\beta_1}\mathscr{D}^\beta\left(\mathfrak{h}x\partial_x[\partial_x u]\right) - x^{\beta_1}\mathfrak{h}\mathscr{D}^\beta\left(x\partial_x[\partial_x u]\right)\|^2_{\mathscr{H}_0^\alpha(\mathbf{H}_{\lambda,\tau})}$$

$$+\|\mathfrak{h}\underbrace{\left\{x^{\beta_1}\mathscr{D}^\beta\left(x\partial_x[\partial_x u]\right) - x^{\beta_1}x\mathscr{D}^\beta\left(\partial_x^2 u\right)\right\}}_{=\ \beta_1 x^{\beta_1}\mathscr{D}^\beta(\partial_x u)}\|^2_{\mathscr{H}_0^\alpha(\mathbf{H}_{\lambda,\tau})}$$

$$\leq\ C_s \Big(\|x\partial_x[\partial_x u]\|^2_{\mathscr{B}_0^\alpha} \|\mathfrak{h}\|^2_{\mathscr{G}_k^0} + \|x\partial_x[\partial_x u]\|^2_{\mathscr{H}_{k-1}^\alpha} \|\mathfrak{h}\|^2_{\mathscr{C}^0_{\{x=0\},1}}$$

$$+\|\mathfrak{h}\|^2_{L^\infty} \|\partial_x u\|^2_{\mathscr{H}_k^\alpha} \Big)$$

$$\leq\ C_s \left(\|\partial_x u\|^2_{\mathscr{B}_1^\alpha} \|\mathfrak{h}\|^2_{\mathscr{G}_k^0} + \|\mathfrak{h}\|^2_{\mathscr{C}^0_{\{x=0\},1}} E^\alpha_{k,\lambda}[u(\tau)] \right). \qquad (5.71)$$

We note that $\|x^j \partial_x^j \Phi\|_{\mathscr{H}_k^\alpha} \leq \|\Phi\|_{\mathscr{H}_{k+j}^\alpha}$ which can be shown by induction. In order to estimate the term A^{00}, we proceed as follows:

$$A^{00} = \left[\mathscr{D}^\beta, \mathfrak{g}^{\tau\tau}\left(\partial_\tau - \partial_x\right)^2\right] u\ =\ \mathscr{D}^\beta\left([-1 + x\mathfrak{h}^0]\left(\partial_\tau - \partial_x\right)^2 u\right) - [-1 + x\mathfrak{h}^0]\mathscr{D}^\beta\left(\partial_\tau - \partial_x\right)^2 u$$

$$=\ \mathscr{D}^\beta\left([x\mathfrak{h}^0]\left(\partial_\tau - \partial_x\right)^2 u\right) - [x\mathfrak{h}^0]\mathscr{D}^\beta\left(\partial_\tau - \partial_x\right)^2 u\,. \qquad (5.72)$$

Now using equation (5.1), (5.60) and (5.27), we obtain the following expression of $\left(\partial_\tau - \partial_x\right)^2 u$:

$$\left(\partial_\tau - \partial_x\right)^2 u\ =\ -2\left(\hat{\mathfrak{g}}^{\tau\tau} + \hat{\mathfrak{g}}^{\tau x}\right)\left(\partial_\tau - \partial_x\right)\partial_x - \left(\hat{\mathfrak{g}}^{\tau\tau} + 2\hat{\mathfrak{g}}^{\tau x} + \hat{\mathfrak{g}}^{xx}\right)\partial_x^2 - 2\hat{\mathfrak{g}}^{\tau A}\left(\partial_\tau - \partial_x\right)\partial_A$$

$$-2\left(\hat{\mathfrak{g}}^{xA} + \hat{\mathfrak{g}}^{\tau A}\right)\partial_x\partial_A - \mathfrak{g}^{AB}\partial_A\partial_B\ -\hat{\Upsilon}^\sigma\partial_\sigma u + \hat{F}\,. \qquad (5.73)$$

Here the hat means multiplication with $1/\mathfrak{g}^{\tau\tau}$ (recall $|\mathfrak{g}^{\tau\tau}| > \epsilon_0 > 0$). We will need the following:

LEMMA 5.9 *Let*

$$\tilde{\partial} = (x\partial_x, \partial_A), \quad k \in \mathbb{N}^*, \quad \theta \in \mathbb{R}, \quad \hat{\psi} = \frac{\psi}{\mathfrak{g}^{\tau\tau}}\,, \quad \left|\frac{1}{\mathfrak{g}^{\tau\tau}}\right| \leq \frac{1}{\epsilon_0}\,. \qquad (5.74)$$

We have the following estimates:

$$\|\hat{\psi}\|_{\mathscr{C}^{\theta}_{\{x=0\},0}} \leq \frac{1}{\epsilon_0}\|\psi\|_{\mathscr{C}^{\theta}_{\{x=0\},0}} , \qquad (5.75)$$

$$\|\hat{\psi}\|_{\mathscr{C}^{\theta}_{\{x=0\},1}} \leq \frac{1}{\epsilon_0}\|\psi\|_{\mathscr{C}^{\theta}_{\{x=0\},1}} + \frac{1}{\epsilon_0^2}\|\tilde{\partial}(x\mathfrak{h}^0)\|_{L^\infty}\|\psi\|_{\mathscr{C}^{\theta}_{\{x=0\},0}} , \qquad (5.76)$$

and

$$\|\hat{\psi}\|_{\mathscr{H}^{\theta}_k} \leq \frac{1}{\epsilon_0}\|\psi\|_{\mathscr{H}^{\theta}_k} + C(\epsilon_0, \|\mathfrak{h}^0\|_{L^\infty})\|\psi\|_{\mathscr{B}^{\theta}_0}\left(1 + \|\mathfrak{h}^0\|_{\mathscr{H}^{-1}_k}\right) , \qquad (5.77)$$

with identical estimates with $\mathscr{C}^{\theta}_{\{x=0\},0}$ replaced by $\mathscr{B}^{\theta}_0$ and $\mathscr{H}^{\theta}_k$ replaced by $\mathscr{G}^{\theta}_k$.

PROOF: The first inequality is obvious. Next:

$$\begin{aligned}
\|\hat{\psi}\|_{\mathscr{C}^{\theta}_{\{x=0\},1}} &\leq \|x^{-\theta}\frac{1}{\mathfrak{g}^{\tau\tau}}\psi\|_{L^\infty} + \|x^{-\theta}\tilde{\partial}\left\{\frac{1}{\mathfrak{g}^{\tau\tau}}\psi\right\}\|_{L^\infty} \\
&\leq \frac{1}{\epsilon_0}\|\psi\|_{\mathscr{C}^{\theta}_{\{x=0\},0}} + \|x^{-\theta}\left\{\psi\tilde{\partial}(\frac{1}{\mathfrak{g}^{\tau\tau}}) + \frac{1}{\mathfrak{g}^{\tau\tau}}\tilde{\partial}\psi\right\}\|_{L^\infty} \\
&\leq \frac{1}{\epsilon_0}\|\psi\|_{\mathscr{C}^{\theta}_{\{x=0\},0}} + \frac{1}{\epsilon_0^2}\|\psi\|_{\mathscr{C}^{\theta}_{\{x=0\},0}}\|\tilde{\partial}(x\mathfrak{h}^0)\|_{L^\infty} + \frac{1}{\epsilon_0}\|\tilde{\partial}\psi\|_{\mathscr{C}^{\theta}_{\{x=0\},0}} \\
&\leq \frac{1}{\epsilon_0}\|\psi\|_{\mathscr{C}^{\theta}_{\{x=0\},1}} + \frac{1}{\epsilon_0^2}\|\tilde{\partial}(x\mathfrak{h}^0)\|_{L^\infty}\|\psi\|_{\mathscr{C}^{\theta}_{\{x=0\},0}} .
\end{aligned}$$

On the other hand, from inequality (A.27) of [13] we have:

$$\begin{aligned}
\|\hat{\psi}\|_{\mathscr{H}^{\theta}_k} &= \|\frac{1}{\mathfrak{g}^{\tau\tau}}\psi\|_{\mathscr{H}^{\theta}_k} \leq \|\psi\|_{\mathscr{B}^{\theta}_0}\|\frac{1}{\mathfrak{g}^{\tau\tau}}\|_{\mathscr{G}^{\theta}_k} + \|\psi\|_{\mathscr{H}^{\theta}_k}\|\frac{1}{\mathfrak{g}^{\tau\tau}}\|_{\mathscr{C}^{0}_{\{x=0\},0}} \\
&\leq \frac{1}{\epsilon_0}\|\psi\|_{\mathscr{H}^{\theta}_k} + \|\psi\|_{\mathscr{B}^{\theta}_0}\|\frac{1}{\mathfrak{g}^{\tau\tau}}\|_{\mathscr{G}^{\theta}_k} . \qquad (5.78)
\end{aligned}$$

Now, by hypothesis we have,

$$\begin{aligned}
\frac{1}{\mathfrak{g}^{\tau\tau}(\tau,x,v^A)} &= \frac{1}{-1 + x\mathfrak{h}^0(\tau,x,v^A)} = -1 + \frac{x\mathfrak{h}^0(\tau,x,v^A)}{-1 + x\mathfrak{h}^0(\tau,x,v^A)} \\
&= -1 + G(\tau,x,v^A,x\mathfrak{h}^0) ,
\end{aligned}$$

where G is any function which takes the correct values in the range of interest, e.g.,

$$G(\tau,x,v^A,p) = \frac{p\chi(p)}{-1+p} \quad \text{with} \quad \chi \in C^\infty(\mathbb{R}) \quad \text{such that} \quad \chi(p) = \begin{cases} 1 & \text{if} \quad p \leq 1 - \frac{3\epsilon_0}{4} \\ 0 & \text{if} \quad p \geq 1 - \frac{\epsilon_0}{4} \end{cases} .$$

Recall that hypothesis (5.2) reads $x\mathfrak{h}^0 \leq 1 - \epsilon_0$. We have (note that the space of functions $\mathscr{G}_k^\theta$ contains constant functions)

$$\|\frac{1}{\mathfrak{g}^{\tau\tau}}\|_{\mathscr{G}_k^0} \leq \|1\|_{\mathscr{G}_k^0} + \|G(.\,,x\mathfrak{h}^0)\|_{\mathscr{G}_k^0} \leq C\left(1 + \|G(.,\,x\mathfrak{h}^0)\|_{\mathscr{G}_k^0}\right) . \qquad (5.79)$$

The function G satisfies the following, for any $p \in \mathbb{R}$:

$$\|G(.\,,p)\|_{\mathscr{C}^0_{\{x=0\},k}} = \|G(.\,,p)\|_{\mathscr{C}^0_{\{x=0\},0}} \leq C(\epsilon_0)$$

and for $i = 0,1$;

$$\left\|\frac{\partial^i G(.\,,p)}{\partial p^i}\right\|_{\mathscr{C}^0_{\{x=0\},k-i}} \leq C(\epsilon_0)|p|^{1-i} .$$

These two inequalities show that G has a uniform zero of order 1 at $p = 0$. Therefore, we can apply inequality (A.31) of [13] and obtain that

$$\|G(.\,,x\mathfrak{h}^0)\|_{\mathscr{G}_k^0} \leq C(\epsilon_0, \|\mathfrak{h}^0\|_{L^\infty})\|h^0\|_{\mathscr{H}_k^{-1}} .$$

This implies (see (5.79))

$$\|\frac{1}{\mathfrak{g}^{\tau\tau}}\|_{\mathscr{G}_k^0} \leq C(\epsilon_0, \|\mathfrak{h}^0\|_{L^\infty})\left(1 + \|\mathfrak{h}^0\|_{\mathscr{H}_k^{-1}}\right), \qquad (5.80)$$

and (5.78) leads to (5.77). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

If we insert (5.73) into equation (5.72), we obtain seven commutators which we label A_a^{00} , $a = 1,\dots,7$. These terms can be estimated in the same way as we did before, using (A.34), (A.35) of [13] and Lemma 5.9. They will be analyzed in the order $7 - 3 - 5 - 1 - 2 - 4 - 6$. Let us estimate the term A_7^{00} containing the source term F. We have

$$
\begin{aligned}
\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1}\left\{A_7^{00}\right\}^2 dx\,d\nu &= \|x^{\beta_1}\mathscr{D}^\beta\left([x\mathfrak{h}^0]\hat{F}\right) - x^{\beta_1}[x\mathfrak{h}^0]\mathscr{D}^\beta\hat{F}\|^2_{\mathscr{H}_0^\alpha} \\[2mm]
&\leq C\left(\|\hat{F}\|^2_{\mathscr{B}_0^\alpha}\|x\mathfrak{h}^0\|^2_{\mathscr{G}_k^0} + \|\hat{F}\|^2_{\mathscr{H}_{k-1}^\alpha}\|x\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\right) \\[2mm]
&\leq C(\epsilon_0)\|F\|^2_{\mathscr{B}_0^\alpha}\|x\mathfrak{h}^0\|^2_{\mathscr{G}_k^0} + C(\epsilon_0)\|x\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}} \\[2mm]
&\quad \times \left\{\|F\|^2_{\mathscr{H}_{k-1}^\alpha} + \|F\|^2_{\mathscr{B}_0^\alpha}C(\|\mathfrak{h}^0\|_{L^\infty})\left(1 + \|x\mathfrak{h}^0\|_{\mathscr{H}_{k-1}^0}\right)\right\}. \quad (5.81)
\end{aligned}
$$

The third term can be estimated as follows:

$$\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1} \left\{ A_3^{00} \right\}^2 dx\, d\nu$$

$$= 2\|\mathscr{D}^\beta \left(x\mathfrak{h}^0 \hat{\mathfrak{g}}^{\tau A} \partial_A (\partial_\tau - \partial_x) u \right) - x\mathfrak{h}^0 \mathscr{D}^\beta \left(\hat{\mathfrak{g}}^{\tau A} \partial_A (\partial_\tau - \partial_x) u \right) \|^2_{\mathscr{H}_0^\alpha}$$

$$\leq C\|\hat{\mathfrak{g}}^{\tau A} \partial_A (\partial_\tau - \partial_x) u\|^2_{\mathscr{B}_0^\alpha} \|x\mathfrak{h}^0\|^2_{\mathscr{G}_k^0} + C\|\hat{\mathfrak{g}}^{\tau A} \partial_A (\partial_\tau - \partial_x) u\|^2_{\mathscr{H}_{k-1}^\alpha} \|x\mathfrak{h}^0\|^2_{\mathscr{C}_{\{x=0\},1}^0}$$

$$\leq C\|\hat{\mathfrak{g}}^\sharp\|^2_{L^\infty} \|(\partial_\tau - \partial_x) u\|^2_{\mathscr{B}_1^\alpha} \|x\mathfrak{h}^0\|^2_{\mathscr{G}_k^0}$$

$$+ C\|x\mathfrak{h}^0\|^2_{\mathscr{C}_{\{x=0\},1}^0} \left\{ \|(\partial_\tau - \partial_\tau) u\|^2_{\mathscr{B}_1^\alpha} \|\hat{\mathfrak{g}}^\sharp\|^2_{\mathscr{G}_{k-1}^0} + \|(\partial_\tau - \partial_x) u\|^2_{\mathscr{H}_k^\alpha} \|\hat{\mathfrak{g}}^\sharp\|^2_{\mathscr{C}_{\{x=0\},0}^0} \right\}$$

$$\leq C(\epsilon_0)\|\mathfrak{g}^\sharp\|^2_{L^\infty} \|(\partial_\tau - \partial_x) u\|^2_{\mathscr{B}_1^\alpha} \|x\mathfrak{h}^0\|^2_{\mathscr{G}_k^0} + C(\epsilon_0)\|x\mathfrak{h}^0\|^2_{\mathscr{C}_{\{x=0\},1}^0} \|\mathfrak{g}^\sharp\|^2_{L^\infty} \|(\partial_\tau - \partial_x) u\|^2_{\mathscr{H}_k^\alpha}$$

$$+ \|x\mathfrak{h}^0\|^2_{\mathscr{C}_{\{x=0\},1}^0} \|(\partial_\tau - \tau) u\|^2_{\mathscr{B}_1^\alpha} \left\{ \|\mathfrak{g}^\sharp\|^2_{\mathscr{G}_{k-1}^0} + \|\mathfrak{g}^\sharp\|^2_{L^\infty} C(\|\mathfrak{h}^0\|_{L^\infty}) \left(1 + \|x\mathfrak{h}^0\|^2_{\mathscr{G}_{k-1}^0} \right) \right\} \, . \tag{5.82}$$

A similar analysis applies to A_5^{00}.

As far as the first term A_1^{00} is concerned, we have

$$-\frac{1}{2} A_1^{00} = \mathscr{D}^\beta \left(x\mathfrak{h}^0 \hat{\mathfrak{h}}^1 (x\partial_x)(\partial_\tau - \partial_x) u \right) - x\mathfrak{h}^0 \mathscr{D}^\beta \left(\hat{\mathfrak{h}}^1 (x\partial_x)(\partial_\tau - \partial_x) u \right) \, .$$

Using again the weighted Moser-type inequality of [13], we can evaluate the square of its norm as follows:

$$\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1} (A_1^{00})^2 dx\, d\nu$$

$$= 2\|x^{\beta_1} \mathscr{D}^\beta \left([x\mathfrak{h}^0][\hat{\mathfrak{h}}^1 (x\partial_x)(\partial_\tau - \partial_x) u] \right) - x^{\beta_1} [x\mathfrak{h}^0] \mathscr{D}^\beta \left(\hat{\mathfrak{h}}^1 (x\partial_x)(\partial_\tau - \partial_x) u \right) \|^2_{\mathscr{H}_0^\alpha}$$

$$\leq C \left(\|\hat{\mathfrak{h}}^1 (x\partial_x)(\partial_\tau - \partial_x) u\|^2_{\mathscr{B}_0^\alpha} \|x\mathfrak{h}^0\|^2_{\mathscr{G}_k^0} + \|\hat{\mathfrak{h}}^1 (x\partial_x)(\partial_\tau - \partial_x) u\|^2_{\mathscr{H}_{k-1}^\alpha} \|x\mathfrak{h}^0\|^2_{\mathscr{C}_{\{x=0\},1}^0} \right)$$

$$\leq C(\epsilon_0) \left(\|\mathfrak{h}^1\|_{L^\infty} \|(\partial_\tau - \partial_x) u\|^2_{\mathscr{B}_1^\alpha} \|x\mathfrak{h}^0\|^2_{\mathscr{G}_k^0} + \|\hat{\mathfrak{h}}^1 (x\partial_x)(\partial_\tau - \partial_x) u\|^2_{\mathscr{H}_{k-1}^\alpha} \|x\mathfrak{h}^0\|^2_{\mathscr{C}_{\{x=0\},1}^0} \right) \, .$$

Using now inequality (A.34) of [13], we have (the last inequality is obtained by using (5.77):

$$\|\hat{\mathfrak{h}}^1(x\partial_x)(\partial_\tau - \partial_x)u\|^2_{\mathscr{H}^\alpha_{k-1}}$$

$$\leq \; C\left(\|x\partial_x(\partial_\tau - \partial_x)u\|^2_{\mathscr{B}^\alpha_0}\|\hat{\mathfrak{h}}^1\|^2_{\mathscr{G}^0_{k-1}} + \|x\partial_x(\partial_\tau - \partial_x)u\|^2_{\mathscr{H}^\alpha_{k-1}}\|\hat{\mathfrak{h}}^1\|^2_{\mathscr{C}^0_0}\right)$$

$$\leq \; C\|(\partial_\tau - \partial_x)u\|^2_{\mathscr{B}^\alpha_1}\|\hat{\mathfrak{h}}^1\|^2_{\mathscr{G}^0_{k-1}} + C(\epsilon_0)\|\mathfrak{h}^1\|^2_{L^\infty}\|(\partial_\tau - \partial_x)u\|^2_{\mathscr{H}^\alpha_k}$$

$$\leq \; C(\epsilon_0)\|\mathfrak{h}^1\|^2_{L^\infty}\|(\partial_\tau - \partial_x)u\|^2_{\mathscr{H}^\alpha_k} + C(\epsilon_0)\|(\partial_\tau - \partial_x)u\|^2_{\mathscr{B}^\alpha_1}$$

$$\times \left\{\|\mathfrak{h}^1\|^2_{\mathscr{G}^0_{k-1}} + C(\|\mathfrak{h}^0\|_{L^\infty})\|\mathfrak{h}^1\|^2_{L^\infty}(1 + \|x\mathfrak{h}^0\|^2_{\mathscr{G}^0_{k-1}})\right\},$$

which gives

$$\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1}(A^{00}_1)^2 dx\, d\nu$$

$$\leq \; C(\epsilon_0)\|\mathfrak{h}^1\|^2_{L^\infty}\|(\partial_\tau - \partial_x)u\|^2_{\mathscr{B}^\alpha_1}\|x\mathfrak{h}^0\|^2_{\mathscr{G}^0_k} + C(\epsilon_0)\|x\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\|\mathfrak{h}^1\|^2_{L^\infty}\|(\partial_\tau - \partial_x)u\|^2_{\mathscr{H}^\alpha_k}$$

$$+C(\epsilon_0)\|x\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\|(\partial_\tau - \partial_x)u\|^2_{\mathscr{B}^\alpha_1}\left\{\|\mathfrak{h}^1\|^2_{\mathscr{G}^0_{k-1}} + C(\|\mathfrak{h}^0\|_{L^\infty})\|\mathfrak{h}^1\|^2_{L^\infty}(1 + \|x\mathfrak{h}^0\|^2_{\mathscr{G}^0_{k-1}})\right\}$$

$$\leq \; C(\epsilon_0)\|x\mathfrak{h}^0\|_{\mathscr{C}^0_{\{x=0\},1}}\|\mathfrak{h}^1\|^2_{L^\infty}\|(\partial_\tau - \partial_x)u\|^2_{\mathscr{H}^\alpha_k}$$

$$+C(\epsilon_0)\left(1 + \|x\mathfrak{h}^0\|_{\mathscr{C}^0_{\{x=0\},1}}\right)\|(\partial_\tau - \partial_x)u\|^2_{\mathscr{B}^\alpha_1}\left\{\|\mathfrak{h}^1\|^2_{\mathscr{G}^0_{k-1}} + C(\|\mathfrak{h}^0\|_{L^\infty})\|\mathfrak{h}^1\|^2_{L^\infty}(1 + \|x\mathfrak{h}^0\|^2_{\mathscr{G}^0_k})\right\}. \tag{5.83}$$

Since the terms A^{00}_1 and A^{00}_4 have same the structure, to estimate the second one, we just have to replace in the estimate on A^{00}_1, $\|(\partial_\tau - \partial_x)u\|^2_{\mathscr{H}^\alpha_k}$ by $\|\partial_x u\|^2_{\mathscr{H}^\alpha_k}$ and $\|x\mathfrak{h}^1\|^2_{\mathscr{G}^0_{k-1}}$ by $\|\hat{\mathfrak{g}}^{\tau A} + \hat{\mathfrak{g}}^{xA}\|^2_{\mathscr{G}^0_{k-1}}$.

We continue with the most dangerous term A^{00}_2. We have (recall that $\hat{1} = 1/\mathfrak{g}^{\tau\tau}$)

$$-A^{00}_2 = \mathscr{D}^\beta\left([x\mathfrak{h}^0](\hat{1} + x\hat{\mathfrak{h}})\partial^2_x u\right) - [x\mathfrak{h}^0]\mathscr{D}^\beta\left([\hat{1} + x\hat{\mathfrak{h}}]\partial^2_x u\right),$$

$$\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1}\left\{A^{00}_2\right\}^2 \; = \; \|x^{\beta_1}\mathscr{D}^\beta\left([\mathfrak{h}^0][\hat{1} + x\hat{\mathfrak{h}}](x\partial_x)\partial_x u\right)$$

$$-x^{\beta_1}[x\mathfrak{h}^0]\mathscr{D}^\beta\left([\hat{1} + x\hat{\mathfrak{h}}]\partial^2_x u\right)\|^2_{\mathscr{H}^\alpha_0}$$

$$\leq \; \|x^{\beta_1}\mathscr{D}^\beta\left(x\mathfrak{h}^0(\hat{1}.\partial^2_x u)\right) - x^{\beta_1}[x\mathfrak{h}^0]\mathscr{D}^\beta\left(\hat{1}.\partial^2_x u\right)\|^2_{\mathscr{H}^\alpha_0}$$

$$+\|x^{\beta_1}\mathscr{D}^\beta\left(x\mathfrak{h}^0\hat{\mathfrak{h}}x\partial_x(\partial_x u)\right) - x^{\beta_1}[x\mathfrak{h}^0]\mathscr{D}^\beta\left(\hat{\mathfrak{h}}x\partial_x(\partial_x u)\right)\|^2_{\mathscr{H}^\alpha_0}$$

$$=: \; (a) + (b).$$

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Now, estimating these two expressions as we did with A_1^{00}, we obtain the following

$$
\begin{aligned}
(b) &\leq C\left(\|\hat{\mathfrak{h}}(x\partial_x)\partial_x u\|^2_{\mathscr{B}_0^\alpha}\|x\mathfrak{h}^0\|^2_{\mathscr{G}_k^0} + \|\hat{\mathfrak{h}}(x\partial_x)\partial_x u\|^2_{\mathscr{H}_{k-1}^\alpha}\|x\mathfrak{h}^0\|^2_{\mathscr{C}_{\{x=0\},1}^0}\right) \\
&\leq C(\epsilon_0)\|\mathfrak{h}\|^2_{L^\infty}\|\partial_x u\|^2_{\mathscr{B}_1^\alpha}\|x\mathfrak{h}^0\|^2_{\mathscr{G}_k^0} + C\|\hat{\mathfrak{h}}(x\partial_x)\partial_x u\|^2_{\mathscr{H}_{k-1}^\alpha}\|x\mathfrak{h}^0\|^2_{\mathscr{C}_{\{x=0\},1}^0} \;.
\end{aligned}
$$

Now equations (A.34) of [13] and (5.77) give

$$
\begin{aligned}
\|\hat{\mathfrak{h}}(x\partial_x)\partial_x u\|^2_{\mathscr{H}_{k-1}^\alpha} &\leq C\left(\|(x\partial_x)\partial_x u\|^2_{\mathscr{B}_0^\alpha}\|\hat{\mathfrak{h}}\|^2_{\mathscr{G}_{k-1}^0} + \|(x\partial_x)\partial_x u\|^2_{\mathscr{H}_{k-1}^\alpha}\|\hat{\mathfrak{h}}\|^2_{\mathscr{C}_{\{x=0\},0}^0}\right) \\
&\leq C(\epsilon_0)\|\mathfrak{h}\|^2_{L^\infty}\|\partial_x u\|^2_{\mathscr{H}_k^\alpha} \\
&\quad + C(\epsilon_0)\|\partial_x u\|^2_{\mathscr{B}_1^\alpha}\left\{\|\mathfrak{h}\|^2_{\mathscr{G}_{k-1}^0} + \|\mathfrak{h}\|^2_{L^\infty}C(\|\mathfrak{h}^0\|_{L^\infty})(1 + \|x\mathfrak{h}^0\|^2_{\mathscr{G}_{k-1}^0})\right\} \;,
\end{aligned}
$$

which gives the desired estimate for (b).

In order to estimate the term (a) we write $\beta = (\beta_1, \beta_v)$ and $\mathscr{D}^\beta = \partial_x^{\beta_1} X_v^{\beta_v}$, with $X_v^{\beta_v} = X_2^{\beta_2} \ldots X_r^{\beta_r}$:

$$
\begin{aligned}
\mathscr{D}^\beta\left(\mathfrak{h}^0 x(\hat{1}.\partial_x^2 u)\right) - [x\mathfrak{h}^0]\mathscr{D}^\beta\left(\hat{1}.\partial_x^2 u\right) &= \mathscr{D}^\beta\left(\mathfrak{h}^0 x(\hat{1}.\partial_x^2 u)\right) - \mathfrak{h}^0\mathscr{D}^\beta\left(\hat{1}.x\partial_x^2 u\right) \\
&\quad + \mathfrak{h}^0\mathscr{D}^\beta\left(\hat{1}.x\partial_x^2 u\right) - [x\mathfrak{h}^0]\mathscr{D}^\beta\left(\hat{1}.\partial_x^2 u\right) \\
&=: (1) + (2) \;. \qquad\qquad (5.84)
\end{aligned}
$$

We have

$$
x^{\beta_1}(2) = \beta_1\mathfrak{h}^0 x^{\beta_1}\partial_v^{\beta'}\partial_x^{\beta_1-1}(\hat{1}.\partial_x^2 u)
$$

as well as

$$
x^{-2\alpha-1+2\beta_1}(2)^2 = \beta_1^2(\mathfrak{h}^0)^2 x^{-2(\alpha-1)-1+2(\beta_1-1)}\left(\partial_v^{\beta'}\partial_x^{\beta_1-1}(\hat{1}.\partial_x^2 u)\right)^2 \;.
$$

This identity leads to

$$
\begin{aligned}
\|x^{\beta_1}(2)\|^2_{\mathscr{H}_0^\alpha} &\leq C\|\mathfrak{h}^0\|^2_{L^\infty}\|\hat{1}.\partial_x^2 u\|^2_{\mathscr{H}_{k-1}^{\alpha-1}} \\
&\leq C\|\mathfrak{h}^0\|^2_{L^\infty}\left(\|\partial_x^2 u\|^2_{\mathscr{B}_0^{\alpha-1}}\|\hat{1}\|^2_{\mathscr{G}_{k-1}^0} + \|\partial_x^2 u\|^2_{\mathscr{H}_{k-1}^{\alpha-1}}\|\hat{1}\|^2_{\mathscr{C}_0^0}\right) \\
&\leq C\|\mathfrak{h}^0\|^2_{L^\infty}\left(\|\partial_x u\|^2_{\mathscr{B}_1^\alpha}\|\hat{1}\|^2_{\mathscr{G}_{k-1}^0} + \frac{1}{\epsilon_0}\|\partial_x u\|^2_{\mathscr{H}_k^\alpha}\right) \;.
\end{aligned}
$$

Using again (5.77) we have:

$$
\|\hat{1}\|^2_{\mathscr{G}_{k-1}^0} \leq \frac{1}{\epsilon_0}\|1\|^2_{\mathscr{G}_{k-1}^0} + C(\|\mathfrak{h}^0\|_{L^\infty})\left(1 + \|x^2\mathfrak{h}^0\|^2_{\mathscr{G}_{k-1}^0}\right),
$$

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that is

$$\|\hat{1}\|^2_{\mathscr{G}^0_{k-1}} \le C(\|\mathfrak{h}^0\|_{L^\infty})\left(1 + \|x^2\mathfrak{h}^0\|^2_{\mathscr{G}^0_{k-1}}\right) . \tag{5.85}$$

Thus,

$$\|x^{\beta_1}(2)\|^2_{\mathscr{H}^\alpha_0} \le C(\|\mathfrak{h}^0\|_{L^\infty})\left\{\|\partial_x u\|^2_{\mathscr{B}^\alpha_1}\left(1 + \|x\mathfrak{h}^0\|^2_{\mathscr{G}^0_{k-1}}\right) + \frac{1}{\epsilon_0}E^\alpha_{k,\lambda}[u(\tau)]\right\} . \tag{5.86}$$

As far as the first term of (5.84) is concerned, we have:

$$\begin{aligned}
\|x^{\beta_1}(1)\|^2_{\mathscr{H}^\alpha_0} &= \|x^{\beta_1}\mathscr{D}^\beta\left(\mathfrak{h}^0(\hat{1}.(x\partial_x)\partial_x u)\right) - x^{\beta_1}\mathfrak{h}^0\mathscr{D}^\beta\left(\hat{1}.(x\partial_x)\partial_x u\right)\|^2_{\mathscr{H}^\alpha_0} \\
&\le C\left\{\|\hat{1}.(x\partial_x)\partial_x u\|^2_{\mathscr{B}^\alpha_0}\|\mathfrak{h}^0\|^2_{\mathscr{G}^0_k} + \|\hat{1}.(x\partial_x)\partial_x u\|^2_{\mathscr{H}^\alpha_{k-1}}\|\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\right\} \\
&\le C(\epsilon_0)\left\{\|\partial_x u\|^2_{\mathscr{B}^\alpha_1}\|\mathfrak{h}^0\|^2_{\mathscr{G}^0_k}\right. \\
&\qquad \left. + \|\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\left\{\|(x\partial_x)\partial_x u\|^2_{\mathscr{B}^\alpha_0}\|\hat{1}\|^2_{\mathscr{G}^0_{k-1}} + \|(x\partial_x)\partial_x u\|^2_{\mathscr{H}^\alpha_{k-1}}\|\hat{1}\|^2_{\mathscr{C}^0_{\{x=0\},0}}\right\}\right\} \\
&\le C(\epsilon_0)\left\{\|\partial_x u\|^2_{\mathscr{B}^\alpha_1}\|\mathfrak{h}^0\|^2_{\mathscr{G}^0_k} + \|\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\left\{\|\partial_x u\|^2_{\mathscr{B}^\alpha_1}\|\hat{1}\|^2_{\mathscr{G}^0_{k-1}} + \frac{1}{\epsilon_0^2}\|\partial_x u\|^2_{\mathscr{H}^\alpha_k}\right\}\right\} \\
&\le C(\epsilon_0)\left\{\|\partial_x u\|^2_{\mathscr{B}^\alpha_1}\|\mathfrak{h}^0\|^2_{\mathscr{G}^0_k} + \|\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}E^\alpha_{k,\lambda}[u(\tau)]\right\} \\
&\quad + C(\epsilon_0)\|\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\|\partial_x u\|^2_{\mathscr{B}^\alpha_1}\left\{C(\|\mathfrak{h}^0\|_{L^\infty})\left(1 + \|x\mathfrak{h}^0\|^2_{\mathscr{G}^0_{k-1}}\right)\right\} .
\end{aligned} \tag{5.87}$$

Equations (5.86) and (5.87) finish the proof of the desired estimate for (a), and hence for A^2_{00}.

Now let us consider the sixth term A^{00}_6 of A^{00}. We have

$$\widehat{\Upsilon}^\mu\partial_\mu = \widehat{\Upsilon}^\tau\left(\partial_\tau - \partial_x\right) + \left(\widehat{\Upsilon}^x + \widehat{\Upsilon}^\tau\right)\partial_x + \widehat{\Upsilon}^A\partial_A,$$

and we decompose A^{00}_6 as

$$A^{00}_6 = a + b + c. \tag{5.88}$$

We have

$$a := \mathscr{D}^\beta\left([x\mathfrak{h}^0]\widehat{\Upsilon}^\tau(\partial_\tau - \partial_x)u\right) - [x\mathfrak{h}^0]\mathscr{D}^\beta\left(\widehat{\Upsilon}^\tau(\partial_\tau - \partial_x)u\right),$$

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and

$$\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1} a^2 dx\, d\nu$$

$$= \quad \|x^{\beta_1}\mathscr{D}^{\beta}\left([x\mathfrak{h}^0]\widehat{\Upsilon}^{\tau}(\partial_{\tau}-\partial_x)u\right) - x^{\beta_1}[x\mathfrak{h}^0]\mathscr{D}^{\beta}\left(\widehat{\Upsilon}^{\tau}(\partial_{\tau}-\partial_x)u\right)\|^2_{\mathscr{H}^{\alpha}_0}$$

$$\leq \quad C\left(\|\widehat{\Upsilon}^{\tau}(\partial_{\tau}-\partial_x)u\|^2_{\mathscr{B}^{\alpha}_0}\|x\mathfrak{h}^0\|^2_{\mathscr{G}^0_k} + \|\widehat{\Upsilon}^{\tau}(\partial_{\tau}-\partial_x)u\|^2_{\mathscr{H}^{\alpha}_{k-1}}\|x\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\right)$$

$$\leq \quad C(\epsilon_0)\|\Upsilon^{\tau}\|^2_{L^{\infty}}\|(\partial_{\tau}-\partial_x)u\|^2_{\mathscr{B}^{\alpha}_0}\|x\mathfrak{h}^0\|^2_{\mathscr{G}^0_k} + \|x\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}$$

$$\times \left\{(\partial_{\tau}-\partial_x)u\|^2_{\mathscr{B}^{\alpha}_0}\|\widehat{\Upsilon}^{\tau}\|^2_{\mathscr{G}^0_{k-1}} + \|(\partial_{\tau}-\partial_x)u\|^2_{\mathscr{H}^{\alpha}_{k-1}}\|\widehat{\Upsilon}^{\tau}\|^2_{\mathscr{C}^0_{\{x=0\},0}}\right\}.$$

Now, from (5.75) and (5.77) we have

$$\|\widehat{\Upsilon}^{\tau}\|_{\mathscr{C}^0_{\{x=0\},0}} \leq C(\epsilon_0)\|\Upsilon^{\tau}\|_{L^{\infty}},$$

and

$$\|\widehat{\Upsilon}^{\tau}\|^2_{\mathscr{G}^0_{k-1}} \leq C(\epsilon_0)\left\{\|\Upsilon^{\tau}\|^2_{\mathscr{G}^0_{k-1}} + \|\Upsilon^{\tau}\|^2_{L^{\infty}}C(\|\mathfrak{h}^0\|_{L^{\infty}})\left(1+\|x\mathfrak{h}^0\|^2_{\mathscr{G}^0_{k-1}}\right)\right\}.$$

Thus

$$\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1} a^2 dx\, d\nu \quad \leq \quad C(\epsilon_0)\|\Upsilon^{\tau}\|^2_{L^{\infty}}\|(\partial_{\tau}-\partial_x)u\|^2_{\mathscr{B}^{\alpha}_0}\|x\mathfrak{h}^0\|^2_{\mathscr{G}^0_k}$$

$$+C(\epsilon_0)\|x\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\|\Upsilon^{\tau}\|^2_{L^{\infty}}\|(\partial_{\tau}-\partial_x)u\|^2_{\mathscr{H}^{\alpha}_{k-1}}$$

$$+C(\|\mathfrak{h}^0\|_{L^{\infty}})\|x\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\|(\partial_{\tau}-\partial_x)u\|^2_{\mathscr{B}^{\alpha}_0}\|\Upsilon^{\tau}\|^2_{\mathscr{G}^0_{k-1}}$$

$$+C(\epsilon_0)\|x\mathfrak{h}^0\|^2_{\mathscr{C}^0_{\{x=0\},1}}\|(\partial_{\tau}-\partial_x)u\|^2_{\mathscr{B}^{\alpha}_0}\|\Upsilon^{\tau}\|^2_{L^{\infty}}\left(1+\|x\mathfrak{h}^0\|^2_{\mathscr{G}^0_{k-1}}\right).$$

$$(5.89)$$

On the other hand,

$$b := \mathscr{D}^{\beta}\left([x\mathfrak{h}^0](\widehat{\Upsilon}^{\tau}+\widehat{\Upsilon}^x)\partial_x u\right) - [x\mathfrak{h}^0]\mathscr{D}^{\beta}\left((\widehat{\Upsilon}^{\tau}+\widehat{\Upsilon}^x)\partial_x u\right)$$

and we have

$$\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1} b^2 \, dx \, d\nu$$

$$= \quad \| x^{\beta_1} \mathscr{D}^\beta \left([x\mathfrak{h}^0](\widehat{\Upsilon}^\tau + \widehat{\Upsilon}^x)\partial_x u \right) - x^{\beta_1} x\mathfrak{h}^0 \mathscr{D}^\beta \left((\widehat{\Upsilon}^\tau + \widehat{\Upsilon}^x)\partial_x u \right) \|^2_{\mathscr{H}_0^\alpha}$$

$$\leq \quad C \| (\widehat{\Upsilon}^\tau + \widehat{\Upsilon}^x)\partial_x u \|^2_{\mathscr{B}_0^\alpha} \| x\mathfrak{h}^0 \|^2_{\mathscr{G}_k^0} + C \| (\widehat{\Upsilon}^\tau + \widehat{\Upsilon}^x)\partial_x u \|^2_{\mathscr{H}_{k-1}^\alpha} \| x\mathfrak{h}^0 \|^2_{\mathscr{C}_{\{x=0\},1}^0}$$

$$\leq \quad C(\epsilon_0) \| \Upsilon^\tau + \Upsilon^x \|^2_{L^\infty} \| \partial_x u \|^2_{\mathscr{B}_0^\alpha} \| x\mathfrak{h}^0 \|^2_{\mathscr{G}_k^0} + C \| x\mathfrak{h}^0 \|^2_{\mathscr{C}_{\{x=0\},1}^0}$$

$$\times \left\{ \| \partial_x u \|^2_{\mathscr{B}_0^\alpha} \| \widehat{\Upsilon}^\tau + \widehat{\Upsilon}^x \|^2_{\mathscr{G}_{k-1}^0} + \| \partial_x u \|^2_{\mathscr{H}_{k-1}^\alpha} \| \widehat{\Upsilon}^\tau + \widehat{\Upsilon}^x \|^2_{\mathscr{C}_{\{x=0\},0}^0} \right\}$$

$$\leq \quad C(\epsilon_0) \| \Upsilon^\tau + \Upsilon^x \|^2_{L^\infty} \| \partial_x u \|^2_{\mathscr{B}_0^\alpha} \| x\mathfrak{h}^0 \|^2_{\mathscr{G}_k^0}$$

$$+ C(\epsilon_0) \| x\mathfrak{h}^0 \|^2_{\mathscr{C}_{\{x=0\},1}^0} \| \Upsilon^\tau + \Upsilon^x \|^2_{L^\infty} E_{k,\lambda}^\alpha[u(\tau)]$$

$$+ C(\| \mathfrak{h}^0 \|_{L^\infty}) \| x\mathfrak{h}^0 \|^2_{\mathscr{C}_{\{x=0\},1}^0} \| \partial_x u \|^2_{\mathscr{B}_0^\alpha}$$

$$\times \left\{ \| \Upsilon^\tau + \Upsilon^x \|^2_{\mathscr{G}_{k-1}^0} + \| \Upsilon^\tau + \Upsilon^x \|^2_{L^\infty} \left(1 + \| x\mathfrak{h}^0 \|^2_{\mathscr{G}_{k-1}^0} \right) \right\} \, .$$

$$(5.90)$$

The same holds for the term

$$c := \mathscr{D}^\beta \left([x\mathfrak{h}^0]\widehat{\Upsilon}^A \partial_A u \right) - [x\mathfrak{h}^0]\mathscr{D}^\beta \left(\widehat{\Upsilon}^A \partial_A u \right)$$

and we have

$$\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1} c^2 \, dx \, d\nu$$

$$\leq \quad C(\epsilon_0) \| \Upsilon^A \|^2_{L^\infty} \| \partial_A u \|^2_{\mathscr{B}_0^\alpha} \| x\mathfrak{h}^0 \|^2_{\mathscr{G}_k^0}$$

$$+ C(\epsilon_0) \| x\mathfrak{h}^0 \|^2_{\mathscr{C}_{\{x=0\},1}^0} \| \Upsilon^A \|^2_{L^\infty} \| \partial_A u \|^2_{\mathscr{H}_{k-1}^\alpha}$$

$$+ C(\| \mathfrak{h}^0 \|_{L^\infty}) \| x\mathfrak{h}^0 \|^2_{\mathscr{C}_{\{x=0\},1}^0} \| \partial_A u \|^2_{\mathscr{B}_0^\alpha} \| \Upsilon^A \|^2_{\mathscr{G}_{k-1}^0}$$

$$+ C(\epsilon_0) \| x\mathfrak{h}^0 \|^2_{\mathscr{C}_{\{x=0\},1}^0} \| \partial_A u \|^2_{\mathscr{B}_0^\alpha} \| \Upsilon^A \|^2_{L^\infty} \left(1 + \| x\mathfrak{h}^0 \|^2_{\mathscr{G}_{k-1}^0} \right) \, .$$

$$(5.91)$$

This provides the right estimate for A_6^{00}, and hence for of A^{00}.

An identical estimate is obtained on the fourth term A_4 of the commutator (5.61). This finishes the estimation of the commutator $[\Box_{\mathfrak{g}}, \mathscr{D}^\beta]u$ appearing in (5.59), and the proof is complete. $\qquad \square$

5.2.4 Conclusion

The proof of the Proposition 5.7 used essentially Stokes's theorem, the weighted Moser type inequalities (A.34) and (A.35) of Proposition A.3 of [13], and the weighted substitution inequality type (A.31) of the same reference. One of the points there is that all the constants appearing in these inequalities are independent of x_2 (recall that the sets M_{x_2,x_1} there corresponds to the sets $\mathbf{H}_{\lambda,\tau}$ here) which is the distance between the boundary of M_{x_2,x_1} and the null hypersurface $\mathscr{N} = \{x = 0\}$. So, in our case, all the constants involved in the proof of the previous proposition are independent of λ. This allows us to take the limit as λ goes to 0 in (5.52) and obtain an identical inequality with $E_{k,\lambda}^\alpha[u(\tau)]$ there replaced with $E_k^\alpha[u(\tau)]$. Therefore we have proved the following:

PROPOSITION 5.10 *Proposition 5.7 remains true with $\lambda = 0$.*

Inequality (5.52) with $\lambda = 0$ is the key in deriving an existence theorem for the Einstein-Maxwell equations with data on a hyperboloid, singular near $\{x = 0\}$. In this case, we will show that all the $\mathscr{H}_k$ and $\mathscr{G}_k$ norms appearing in this inequality are controlled by the energy.

It turns out that the proof, in Section 7, of global polyhomogeneity of the geodesically complete metrics constructed by Loizelet requires a slightly different inequality. For this we need to split the metric into two parts as

$$\mathfrak{g}^{\alpha\beta} = \mathring{\mathfrak{g}}^{\alpha\beta} + \delta\mathfrak{g}^{\alpha\beta} \ . \tag{5.92}$$

The rationale behind such a splitting is, that the Lorentzian metric $\mathring{\mathfrak{g}}$ will be fixed (in fact, it will be the flat Minkowski metric in our applications), while the correction $\delta\mathfrak{g}$ will eventually depend on the fields. This leads to the obvious corresponding decomposition of Υ,

$$\Upsilon^\alpha = \mathring{\Upsilon}^\alpha + \delta\Upsilon^\alpha \ . \tag{5.93}$$

We assume that there exist constants σ, M and N such that for $\tau \in [\tau_0, \tau_1]$ we have

$$
\begin{aligned}
M \ \geq \ & \|(\mathring{\mathfrak{h}}^\sharp, \mathring{\mathfrak{g}}^\sharp, \mathring{\Upsilon})\|_{\mathscr{G}_k^0(\mathbf{H}_\tau)} + \|(\delta\mathfrak{h}^\sharp, \delta\mathfrak{g}^\sharp, \delta\Upsilon)\|_{\mathscr{C}_{\{x=0\},1}^0(\mathbf{H}_\tau)} \\
& + \|(\partial_x - \partial_\tau)\mathfrak{g}^\sharp\|_{L^\infty(\mathbf{H}_\tau)} \ , \tag{5.94} \\
N \ \geq \ & \|(\partial_\tau u, \partial_x u, \partial_A u)\|_{\mathscr{B}_1^\sigma(\mathbf{H}_\tau)} + \|(\mathfrak{g}^\sharp, \Upsilon)\|_{L^\infty(\mathbf{H}_\tau)} \\
& + \|(\mathring{\mathfrak{g}}^{\tau A}, \mathring{\mathfrak{g}}^{xA})\|_{\mathscr{G}_{k-1}^{\alpha-\sigma}(\mathbf{H}_\tau)} + \|(\delta\mathfrak{g}^\sharp, \delta\Upsilon)\|_{\mathscr{C}_{\{x=0\},1}^0(\mathbf{H}_\tau)} \ . \tag{5.95}
\end{aligned}
$$

We then have:

PROPOSITION 5.11 *Let $k > n/2+1$, $\sigma \in \mathbb{R}$, $\alpha \leq -1/2$. There exist functions $C_3(\epsilon_0, C_0, \alpha, k, n, M)$ and $C_4(\epsilon_0, C_0, \alpha, \sigma, k, n, N)$, monotonously increasing in M and N, which we write as $C_3(M)$ and $C_4(N)$, such that for all*

$$\tau \in [\tau_0, \tau_1]$$

and for all u satisfying (5.1) we have

$$
\begin{aligned}
E_k^\alpha[u(\tau)] \quad \leq \quad & E_k^\alpha[u(\tau_0)] + \int_{\tau_0}^\tau \left\{ C_3(M) \left(E_k^\alpha[u(s)] + \|F(s)\|_{\mathcal{H}_k^\alpha(\mathbf{H}_\tau)}^2 \right) \right. \\
& \left. + C_4(N) \left(1 + \|(\delta \mathfrak{g}^\sharp, \delta \mathfrak{h}^\sharp, \delta \Upsilon)\|_{\mathcal{G}_k^{\alpha-\sigma}(\mathbf{H}_\tau)}^2 \right) \right\} ds \; .
\end{aligned}
\tag{5.96}
$$

PROOF: The result is obtained by calculations very similar to those of Proposition 5.10. We follow that proof until (5.60), which is rewritten as

$$\Box_{\mathfrak{g}} = \mathring{\mathfrak{g}}^{\mu\nu} \partial_{\mu\nu}^2 + \delta \mathfrak{g}^{\mu\nu} \partial_{\mu\nu}^2 + \mathring{\Upsilon}^\nu \partial_\nu + \delta \Upsilon^\nu \partial_\nu \; . \tag{5.97}$$

This leads to the following rewriting of (5.61):

$$
\begin{aligned}
[\Box_{\mathfrak{g}}, \mathscr{D}^\beta] u \quad = \quad & \underbrace{\mathring{\mathfrak{g}}^{\alpha\mu} [\partial_\alpha \partial_\mu, \mathscr{D}^\beta] u}_{=:\mathring{A}_1} + \underbrace{\delta \mathfrak{g}^{\alpha\mu} [\partial_\alpha \partial_\mu, \mathscr{D}^\beta] u}_{=:\delta A_1} \\
& \underbrace{- \mathring{\Upsilon}^\nu [\mathscr{D}^\beta, \partial_\nu] u}_{=:\mathring{A}_2} \underbrace{- \left\{ \mathscr{D}^\beta \left(\mathring{\Upsilon}^\nu \partial_\nu u \right) - \mathring{\Upsilon}^\nu \mathscr{D}^\beta (\partial_\nu u) \right\}}_{=:\mathring{A}_3} \\
& \underbrace{- \delta \Upsilon^\nu [\mathscr{D}^\beta, \partial_\nu] u}_{=:\delta A_2} \underbrace{- \left\{ \mathscr{D}^\beta (\delta \Upsilon^\nu \partial_\nu u) - \delta \Upsilon^\nu \mathscr{D}^\beta (\partial_\nu u) \right\}}_{=:\delta A_3} \\
& \underbrace{- \left\{ \mathscr{D}^\beta (\mathring{\mathfrak{g}}^{\alpha\mu} \partial_\alpha \partial_\mu u) - \mathring{\mathfrak{g}}^{\alpha\mu} \mathscr{D}^\beta (\partial_\alpha \partial_\mu u) \right\}}_{=:\mathring{A}_4} \\
& \underbrace{- \left\{ \mathscr{D}^\beta (\delta \mathfrak{g}^{\alpha\mu} \partial_\alpha \partial_\mu u) - \delta \mathfrak{g}^{\alpha\mu} \mathscr{D}^\beta (\partial_\alpha \partial_\mu u) \right\}}_{=:\delta A_4} \; .
\end{aligned}
\tag{5.98}
$$

The terms $A_i := \mathring{A}_i + \delta A_i$, i=1,2 are estimated as in (5.62)-(5.63). For $\mathring{A}_3$, instead of (5.64) the estimates proceed as before, except that at the end one

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invokes the weighted Sobolev embedding [13, Proposition A.1]; e.g.,

$$
\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1} \left\{ \mathring{I} \right\}^2 dx\, d\nu
$$

$$
= \ \left\| x^{\beta_1} \mathscr{D}^\beta \left(\mathring{\Upsilon}^\tau (\partial_\tau - \partial_x) u \right) - x^{\beta_1} \mathring{\Upsilon}^\tau \mathscr{D}^\beta ((\partial_\tau - \partial_x) u) \right\|^2_{\mathscr{H}^\alpha_0(\mathbf{H}_{\lambda,\tau})}
$$

$$
\leq \ C_s \left(\|(\partial_\tau - \partial_x) u\|^2_{\mathscr{B}^\alpha_0} \|\mathring{\Upsilon}^\tau\|^2_{\mathscr{G}^0_k} + \|(\partial_\tau - \partial_x) u\|^2_{\mathscr{H}^\alpha_{k-1}} \|\mathring{\Upsilon}^\tau\|^2_{\mathscr{C}^0_{\{x=0\},1}} \right)
$$

$$
\leq \ C \left(\|\mathring{\Upsilon}^\tau\|^2_{\mathscr{G}^0_k} + \|\mathring{\Upsilon}^\tau\|^2_{\mathscr{C}^0_{\{x=0\},1}} \right) E^\alpha_k[u(\tau)] . \tag{5.99}
$$

For δA_3, the following version of the inequalities of [13, Proposition A.3] should be used, for any α, β, α' (the proof is identical to that given there):

$$
\|fg\|_{\mathscr{H}^{\alpha+\beta}_k} \leq C \big(\|f\|_{\mathscr{B}^\alpha_{\{x=0\},0}} \|g\|_{\mathscr{G}^\beta_k} + \|g\|_{\mathscr{C}^{\alpha'}_{\{x=0\},0}} \|f\|_{\mathscr{H}^{\alpha+\beta-\alpha'}_k} \big) , \tag{5.100}
$$

and, for $\beta = (\beta_1, \beta')$,

$$
\|x^{\beta_1} \mathscr{D}^\beta (fg) - (x^{\beta_1} \mathscr{D}^\beta f) g)\|_{\mathscr{H}^{\alpha+\beta}_0}
$$

$$
\leq \ C \big(\|f\|_{\mathscr{B}^\alpha_{\{x=0\},0}} \|g\|_{\mathscr{G}^\beta_k} + \|(x\partial_x g, \partial_A g)\|_{\mathscr{C}^{\alpha'}_{\{x=0\},0}} \|f\|_{\mathscr{H}^{\alpha+\beta-\alpha'}_{k-1}} \big) . \tag{5.101}
$$

Instead of (5.64) we then have

$$
\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1} \{\delta I\}^2 dx\, d\nu
$$

$$
= \|x^{\beta_1} \mathscr{D}^\beta (\delta\Upsilon^\tau (\partial_\tau - \partial_x) u) - x^{\beta_1} \delta\Upsilon^\tau \mathscr{D}^\beta ((\partial_\tau - \partial_x) u)\|^2_{\mathscr{H}^\alpha_0(\mathbf{H}_{\lambda,\tau})}
$$

$$
\leq C_s \left(\|(\partial_\tau - \partial_x) u\|^2_{\mathscr{B}^\sigma_0} \|\delta\Upsilon^\tau\|^2_{\mathscr{G}^{\alpha-\sigma}_k} + \|(\partial_\tau - \partial_x) u\|^2_{\mathscr{H}^\alpha_{k-1}} \|\delta\Upsilon^\tau\|^2_{\mathscr{C}^0_{\{x=0\},1}} \right)
$$

$$
\leq C \left(\|(\partial_\tau - \partial_x) u\|^2_{\mathscr{B}^\sigma_0} \|\delta\Upsilon^\tau\|^2_{\mathscr{G}^{\alpha-\sigma}_k} + \|\delta\Upsilon^\tau\|^2_{\mathscr{C}^0_{\{x=0\},1}} E^\alpha_k[u(\tau)] \right) . \tag{5.102}
$$

An identical treatment applies to the remaining three displayed equations following (5.64).

The term A_4 is split into $A^{\mu\nu}$'s as in (5.66), and then for $\mu\nu \neq 00$ we split $A^{\mu\nu} = \mathring{A}^{\mu\nu} + \delta A^{\mu\nu}$ in the obvious way. All the terms $\mathring{A}^{\mu\nu}$ with $\mu\nu \neq 00$ are then treated as in the proof of Proposition 5.10, and at the end we invoke the inequality, for $k \geq n/2 + 1$,

$$
\|f\|^2_{\mathscr{B}^\alpha_1} \leq C \|f\|^2_{\mathscr{H}^\alpha_k} .
$$

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The terms involving $\delta A^{\mu\nu}$ with $\mu\nu \neq 00$ are treated as in (5.102); for example, (5.67) becomes

$$\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1} \left\{\delta A^{AB}\right\}^2 dx\, d\nu$$

$$= \|x^{\beta_1}\mathscr{D}^{\beta}\left(\delta\mathfrak{g}^{AB}\partial_A\partial_B u\right) - x^{\beta_1}\delta\mathfrak{g}^{AB}\mathscr{D}^{\beta}\left(\partial_A\partial_B u\right)\|_{\mathscr{H}_0^{\alpha}(\mathbf{H}_{\lambda,\tau})}$$

$$\leq C_s \sum_A \left(\|\partial_A u\|^2_{\mathscr{B}_1^{\alpha-\sigma}}\|\delta\mathfrak{g}^{\sharp}\|^2_{\mathscr{G}_k^{\sigma}} + \|\partial_A u\|^2_{\mathscr{H}_k^{\alpha}}\|\delta\mathfrak{g}^{\sharp}\|^2_{\mathscr{C}_{\{x=0\},1}^0}\right)$$

$$\leq C \left(\sum_A \|\partial_A u\|^2_{\mathscr{B}_1^{\alpha-\sigma}}\|\delta\mathfrak{g}^{\sharp}\|^2_{\mathscr{G}_k^{\sigma}} + \|\delta\mathfrak{g}^{\sharp}\|^2_{\mathscr{C}_{\{x=0\},1}^0} E_{k,\lambda}^{\alpha}[u(\tau)]\right). \quad (5.103)$$

In (5.82) it is convenient to use the splitting $\mathfrak{h} = \mathring{\mathfrak{h}} + \delta\mathfrak{h}$. The terms involving $\mathring{\mathfrak{h}}$ are estimated, using the Sobolev embedding, by $E_k^{\alpha}[u(\tau)]$, while for those involving $\delta\mathfrak{h}$ we write

$$\int_{\mathbf{H}_{\lambda,\tau}} x^{-2\alpha-1+2\beta_1} \left\{\delta A_3^{00}\right\}^2 dx\, d\nu$$

$$= 2\|\mathscr{D}^{\beta}\left(x\delta\mathfrak{h}^0\hat{\mathfrak{g}}^{\tau A}\partial_A(\partial_\tau - \partial_x)u\right) - x\delta\mathfrak{h}^0\mathscr{D}^{\beta}\left(\hat{\mathfrak{g}}^{\tau A}\partial_A(\partial_\tau - \partial_x)u\right)\|^2_{\mathscr{H}_0^{\alpha}}$$

$$\leq C\|\hat{\mathfrak{g}}^{\tau A}\partial_A(\partial_\tau - \partial_x)u\|^2_{\mathscr{B}_0^{\sigma}}\|x\delta\mathfrak{h}^0\|^2_{\mathscr{G}_k^{\alpha-\sigma}} + C\|\hat{\mathfrak{g}}^{\tau A}\partial_A(\partial_\tau - \partial_x)u\|^2_{\mathscr{H}_{k-1}^{\alpha}}\|x\delta\mathfrak{h}^0\|^2_{\mathscr{C}_{\{x=0\},1}^0}.$$

$$(5.104)$$

The first line above is estimated as

$$C\|\hat{\mathfrak{g}}^{\sharp}\|^2_{L^{\infty}}\|(\partial_\tau - \partial_x)u\|^2_{\mathscr{B}_1^{\sigma}}\|\delta\mathfrak{g}^{\sharp}\|^2_{\mathscr{G}_k^{\alpha-\sigma}},$$

as desired. The second is estimated as

$$C\|\delta\mathfrak{g}^{\sharp}\|^2_{\mathscr{C}_{\{x=0\},1}^0}\left\{\|(\partial_\tau - \partial_\tau)u\|^2_{\mathscr{B}_1^{\sigma}}\|\hat{\mathfrak{g}}^{\tau A}\|^2_{\mathscr{G}_{k-1}^{\alpha-\sigma}} + \|(\partial_\tau - \partial_x)u\|^2_{\mathscr{H}_k^{\alpha}}\|\hat{\mathfrak{g}}^{\sharp}\|^2_{\mathscr{C}_{\{x=0\},0}^0}\right\}$$

$$\leq C\|\delta\mathfrak{g}^{\sharp}\|^2_{\mathscr{C}_{\{x=0\},1}^0}\left\{\|(\partial_\tau - \partial_\tau)u\|^2_{\mathscr{B}_1^{\sigma}}\|\hat{\mathfrak{g}}^{\tau A}\|^2_{\mathscr{G}_{k-1}^{\alpha-\sigma}} + \|\mathfrak{g}^{\sharp}\|^2_{\mathscr{C}_{\{x=0\},0}^0} E_{k,\lambda}^{\alpha}[u(\tau)]\right\}.$$

$$(5.105)$$

To estimate the term A_5^{00} (compare (5.83)) we need to split both $\mathfrak{h}^{\sharp}$ and $\mathfrak{g}^{\sharp}$ into two. The terms there involving $\mathring{\mathfrak{h}}^{\sharp}$ and $\mathring{\mathfrak{g}}^{\sharp}$ can be estimated by $E_k^{\alpha}[u(\tau)]$. The terms involving $\delta\mathfrak{g}^{\sharp}$ are estimated as in the analysis of δA_3^{00}. The mixed

term involving $\mathring{\hat{\mathfrak{g}}}^\sharp$ and $\delta\mathfrak{h}$ is handled in the obvious way

$$\|x^{\beta_1}\mathscr{D}^\beta\left([x\delta\mathfrak{h}^0][\mathring{\hat{\mathfrak{g}}}^{AB}\partial_A\partial_B u]\right) - x^{\beta_1}[x\delta\mathfrak{h}^0]\mathscr{D}^\beta\left([\mathring{\hat{\mathfrak{g}}}^{AB}\partial_A\partial_B u]\right)\|^2_{\mathscr{H}^\alpha_0}$$

$$\leq \ C\big(\|\mathring{\hat{\mathfrak{g}}}^{AB}\partial_A\partial_B u\|_{\mathscr{B}^\sigma_0}\|\|x\delta\mathfrak{h}^0\|_{\mathscr{G}^{\alpha-\sigma}_k} + \|x\delta\mathfrak{h}^0\|_{\mathscr{C}^0_{\{x=0\},1}}\|\mathring{\hat{\mathfrak{g}}}^{AB}\partial_A\partial_B u\|_{\mathscr{H}^\alpha_{k-1}}\big)$$

$$\leq \ C\big(\|\mathring{\hat{\mathfrak{g}}}^\sharp\|_{L^\infty}\|\partial_A u\|_{\mathscr{B}^\sigma_1}\|x\delta\mathfrak{h}^0\|_{\mathscr{G}^{\alpha-\sigma}_k} + \|\mathring{\hat{\mathfrak{g}}}^\sharp\|_{\mathscr{G}^0_{k-1}}\|\partial_A u\|_{\mathscr{H}^\alpha_k}\|x\delta\mathfrak{h}^0\|_{\mathscr{C}^0_{\{x=0\},1}}\big)$$

$$\leq \ C\big(\|\mathring{\hat{\mathfrak{g}}}^\sharp\|_{L^\infty}\|\partial_A u\|_{\mathscr{B}^\sigma_1}\|\delta\mathfrak{g}^\sharp\|_{\mathscr{G}^{\alpha-\sigma}_k} + \|\mathring{\hat{\mathfrak{g}}}^\sharp\|_{\mathscr{G}^0_{k-1}}\|\delta\mathfrak{g}^\sharp\|_{\mathscr{C}^0_1}E^\alpha_{k,\lambda}[u(\tau)]\big)\ . \quad (5.106)$$

A similar analysis of the remaining terms proves the proposition. □

6 Application to the Einstein-Maxwell Equations in wave coordinates and Lorenz gauge

6.1 Change of coordinates

6.1.1 On the gauge condition

Throughout this section, the (unphysical) conformally rescaled metric is denoted by $\mathfrak{g}$, and the (physical) metric is denoted by g; thus $\mathfrak{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$.

Remember that in the original system of coordinates (x^μ) we have

$$\Box_g x^\mu = 0 \qquad \text{with} \qquad g = \eta + h \ ,$$

which leads to

$$\partial_\mu\big(g^{\mu\nu}\sqrt{|\det g|}\big) = 0 \ . \quad (6.1)$$

We want to rewrite the above equation in the new system of coordinate (y^α) (see (6.4)). We have

$$\sqrt{|\det g|} = 1 + \frac{1}{2}\eta^{\alpha\beta}h_{\alpha\beta} + Q(h) \ ,$$

where Q has a uniform zero of order two in h. We set

$$g^{\mu\nu} = \eta^{\mu\nu} + H^{\mu\nu} \ . \quad (6.2)$$

In what follows, we use a generic symbol Q for functions which have a uniform zero of order two. We have

$$\partial_\mu\big(g^{\mu\nu}\sqrt{|\det g|}\big) \ = \ \partial_\mu\big[g^{\mu\nu}\{1 + \frac{1}{2}\eta^{\alpha\beta}h_{\alpha\beta} + Q(h)\}\big]$$

$$= \ \partial_\mu\big[\{\eta^{\mu\nu} + H^{\mu\nu}\}\{1 + \frac{1}{2}\eta^{\alpha\beta}h_{\alpha\beta} + Q(h)\}\big]$$

$$= \ \partial_\mu H^{\mu\nu}\{1 + \frac{1}{2}\eta^{\alpha\beta}h_{\alpha\beta} + Q(h)\} + \{\eta^{\mu\nu} + H^{\mu\nu}\}\{\frac{1}{2}\eta^{\alpha\beta}\partial_\mu h_{\alpha\beta} + \partial_\mu Q(h)\} \ .$$

Using this identity, equation (6.1) takes the form:

$$\partial_\mu H^{\mu\nu} + \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}\partial_\mu h_{\alpha\beta} = -\partial_\mu H^{\mu\nu}\{\frac{1}{2}\eta^{\alpha\beta}h_{\alpha\beta}+Q(h)\}-H^{\mu\nu}\{\frac{1}{2}\eta^{\alpha\beta}\partial_\mu h_{\alpha\beta}+\partial_\mu Q(h)\}-\eta^{\mu\nu}\partial_\mu Q(h) \; .$$
(6.3)

Let us rewrite this equation in the system of coordinates (τ, x, v^A) where

$$y^\mu = \frac{x^\mu}{\eta_{\alpha\beta}x^\alpha x^\beta}, \quad \tau = y^0 \le 0, \quad x = -y^0 - \rho \ge 0 \quad \text{and} \quad y^i = \rho\omega^i(v^A) \; .$$
(6.4)

Recall that

$$\Omega = -y_\alpha y^\alpha = \tau^2 - \rho^2 = x(-\tau + \rho) \ge 0 \; ,$$
(6.5)

and $\hat{f} = \Omega^{-\frac{n-1}{2}}f$ (*not* to be confused with division by $\mathfrak{g}^{\tau\tau}$, as used in the previous section), so that

$$\frac{\partial f}{\partial x^\mu} = \Omega^{\frac{n-1}{2}}\Big\{ -(n-1)y_\mu - \Omega\frac{\partial}{\partial y^\mu} - 2y_\mu y^\alpha\frac{\partial}{\partial y^\alpha}\Big\}\hat{f} \; ,$$
(6.6)

thus the left-hand-side of (6.3) can be rewritten as

$$-(n-1)\Omega^{\frac{n-1}{2}}y_\mu\Big(\widehat{H}^{\mu\nu}+\frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}\hat{h}_{\alpha\beta}\Big)-\Omega^{\frac{n-1}{2}}\Big\{\Omega\frac{\partial}{\partial y^\mu}+2y_\mu y^\alpha\frac{\partial}{\partial y^\alpha}\Big\}\Big(\widehat{H}^{\mu\nu}+\frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}\hat{h}_{\alpha\beta}\Big).$$

We want to analyze the structure of the right-hand side of (6.3). This expression is made of three terms which will be labeled R_1, R_2, and R_3. We have (see (6.6) and recall that $y^\alpha\frac{\partial\Omega}{\partial y^\alpha} = 2\Omega$):

$$\begin{aligned}
R_1 &= \Omega^{\frac{n-1}{2}}\Big\{\frac{1}{2}\Omega^{\frac{n-1}{2}}\mathrm{tr}_\eta(\hat{h}) + Q(\Omega^{\frac{n-1}{2}}\hat{h})\Big\}\Big\{(n-1)y_\mu + \Omega\frac{\partial}{\partial y^\mu} + 2y_\mu y^\alpha\frac{\partial}{\partial y^\alpha}\Big\}\widehat{H}^{\mu\nu} \\
&=: Q(\Omega^{\frac{n-1}{2}}\hat{h}, \Omega^{\frac{n-1}{2}}y_\mu\widehat{H}^{\mu\nu}) + Q(\Omega^{\frac{n-1}{2}}\hat{h}, \Omega^{\frac{n-1}{2}}\partial_\mu\widehat{H}^{\mu\nu}) \\
&\quad +Q(\Omega^{\frac{n-1}{2}}\hat{h}, \Omega^{\frac{n-1}{2}}y_\mu y^\alpha\frac{\partial}{\partial y^\alpha}\widehat{H}^{\mu\nu}) \; .
\end{aligned}$$
(6.7)

Now, since $\frac{\partial Q}{\partial h}$ has a uniform zero of order one, we have

$$\begin{aligned}
\frac{\partial}{\partial x^\mu}Q(h) &= \frac{\partial Q}{\partial h}\frac{\partial h}{\partial x^\mu} = -\frac{\partial Q}{\partial h}(\Omega^{\frac{n-1}{2}}\hat{h})\Omega^{\frac{n-1}{2}}\Big\{(n-1)y_\mu + \Omega\frac{\partial}{\partial y^\mu} + 2y_\mu y^\alpha\frac{\partial}{\partial y^\alpha}\Big\}\hat{h} \\
&=: Q(\Omega^{\frac{n-1}{2}}\hat{h}, \Omega^{\frac{n-1}{2}}y_\mu\hat{h}) + Q(\Omega^{\frac{n-1}{2}}\hat{h}, \Omega^{\frac{n+1}{2}}\frac{\partial\hat{h}}{\partial y^\mu}) + Q(\Omega^{\frac{n-1}{2}}\hat{h}, \Omega^{\frac{n-1}{2}}y_\mu y^\alpha\frac{\partial}{\partial y^\alpha}\hat{h}) \; .
\end{aligned}$$

Thus R_2 reads:

$$
\begin{aligned}
R_2 \;=\; & \frac{1}{2}\eta^{\alpha\beta}\Omega^{\frac{n-1}{2}}\widehat{H}^{\mu\nu}\left\{\Omega^{\frac{n-1}{2}}\right\}\left\{(n-1)y_\mu + \Omega\frac{\partial}{\partial y^\mu} + 2y_\mu y^\alpha\frac{\partial}{\partial y^\alpha}\right\}\hat{h}_{\alpha\beta} \\
& +\Omega^{\frac{n-1}{2}}\widehat{H}^{\mu\nu}\left\{Q(\Omega^{\frac{n-1}{2}}\hat{h},\Omega^{\frac{n-1}{2}}y_\mu\hat{h}) + Q(\Omega^{\frac{n-1}{2}}\hat{h},\Omega^{\frac{n+1}{2}}\partial_\mu\hat{h})\right.\\
& \left. +Q(\Omega^{\frac{n-1}{2}}\hat{h},\Omega^{\frac{n-1}{2}}y_\mu y^\alpha\frac{\partial}{\partial y^\alpha}\hat{h})\right\} \;. \\
=: \;& Q(\Omega^{\frac{n-1}{2}}\hat{h},\Omega^{\frac{n-1}{2}}y_\mu\widehat{H}^{\mu\nu}) + Q(\Omega^{\frac{n-1}{2}}\widehat{H}^{\mu\nu},\Omega^{\frac{n+1}{2}}\partial_\mu\hat{h}) + Q(\Omega^{\frac{n-1}{2}}y_\mu\widehat{H}^{\mu\nu},\Omega^{\frac{n-1}{2}}y^\alpha\frac{\partial}{\partial y^\alpha}\hat{h}) \;.
\end{aligned}
$$
$$(6.8)$$

Next

$$
\begin{aligned}
R_3 \;=\; & -\eta^{\mu\nu}\partial_\mu Q(h) \\
=\; & \eta^{\mu\nu}\left\{Q(\Omega^{\frac{n-1}{2}}\hat{h},\Omega^{\frac{n-1}{2}}y_\mu\hat{h}) + Q(\Omega^{\frac{n-1}{2}}\hat{h},\Omega^{\frac{n+1}{2}}\partial_\mu\hat{h}) + Q(\Omega^{\frac{n-1}{2}}\hat{h},\Omega^{\frac{n-1}{2}}y_\mu y^\alpha\frac{\partial}{\partial y^\alpha}\hat{h})\right\} \;.(6.9)
\end{aligned}
$$

From this, we obtain the following form of the gauge condition (6.3):

$$
\begin{aligned}
y_\mu\widehat{H}^{\mu\nu} + \frac{1}{2}y^\nu\eta^{\alpha\beta}\hat{h}_{\alpha\beta} \;=\; & \frac{1}{1-n}\left\{\Omega\frac{\partial}{\partial y^\mu} + 2y_\mu y^\alpha\frac{\partial}{\partial y^\alpha}\right\}\left(\widehat{H}^{\mu\nu} + \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}\hat{h}_{\alpha\beta}\right) \\
& + \Omega^{-\frac{n-1}{2}}(R_1 + R_2 + R_3) \;.
\end{aligned}
$$
$$(6.10)$$

Now we recall that

$$
H^{\mu\nu} := g^{\mu\nu} - \eta^{\mu\nu} = -h^{\mu\nu} + \widetilde{Q}^{\mu\nu}(h) \;,
$$

where $h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$. Therefore

$$
\eta^{\alpha\beta}\hat{h}_{\alpha\beta} = -\eta_{\alpha\beta}\widehat{H}^{\alpha\beta} + \Omega^{-\frac{n-1}{2}}\widetilde{Q}(\Omega^{\frac{n-1}{2}}\widehat{H}) \;.
$$

Equations (6.7)-(6.10) lead finally to the following form of the gauge condition (6.3):

$$
\begin{aligned}
y_\mu\widehat{H}^{\mu\nu} - \frac{1}{2}y^\nu\mathrm{tr}_\eta(\widehat{H}) \;=\; & \frac{1}{1-n}\left\{\Omega\frac{\partial}{\partial y^\mu} + 2y_\mu y^\alpha\frac{\partial}{\partial y^\alpha}\right\}\left(\widehat{H}^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\mathrm{tr}_\eta\widehat{H}\right) \\
& +\Omega^{-\frac{n-1}{2}}Q(\Omega^{\frac{n-1}{2}}\widehat{H},\Omega^{\frac{n-1}{2}}\widehat{H}) \\
& +\Omega^{-\frac{n-1}{2}}Q(\Omega^{\frac{n-1}{2}}\widehat{H},\Omega^{\frac{n+1}{2}}\partial\widehat{H}) \\
& +\Omega^{-\frac{n-1}{2}}Q(\Omega^{\frac{n-1}{2}}\widehat{H},\Omega^{\frac{n-1}{2}}y^\alpha\frac{\partial}{\partial y^\alpha}\widehat{H}) \;.
\end{aligned}
$$
$$(6.11)$$

We will need the following consequence of this equation: multiplying by y_ν and commuting derivatives one is led to

$$
\begin{aligned}
(n-5)y_\nu y_\mu \widehat{H}^{\mu\nu} \;=\;& 2y^\alpha \frac{\partial}{\partial y^\alpha}\left(y_\mu y_\nu \widehat{H}^{\mu\nu} + \frac{1}{2}\Omega \mathrm{tr}_\eta \widehat{H}\right) \\
&+ \Omega\left(\frac{n-5}{2}\mathrm{tr}_\eta(\widehat{H}) + y_\nu \frac{\partial}{\partial y^\mu}\left(\widehat{H}^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\mathrm{tr}_\eta(\widehat{H})\right)\right) \\
&+ \Omega^{-\frac{n-1}{2}}Q(\Omega^{\frac{n-1}{2}}\widehat{H}, \Omega^{\frac{n-1}{2}}\widehat{H}) \\
&+ \Omega^{-\frac{n-1}{2}}Q(\Omega^{\frac{n-1}{2}}\widehat{H}, \Omega^{\frac{n+1}{2}}\partial\widehat{H}) \\
&+ \Omega^{-\frac{n-1}{2}}Q(\Omega^{\frac{n-1}{2}}\widehat{H}, \Omega^{\frac{n-1}{2}}y^\alpha \frac{\partial}{\partial y^\alpha}\widehat{H}) \,.
\end{aligned}
\tag{6.12}
$$

6.1.2   On the wave equation

In wave coordinates (x^μ), we consider the following wave equation

$$
\eta^{\alpha\beta}\frac{\partial^2 f}{\partial x^\alpha \partial x^\beta} + H^{\alpha\beta}(f, \partial f)\frac{\partial^2 f}{\partial x^\alpha \partial x^\beta} = F(f, \partial f) \,.
\tag{6.13}
$$

In order to check all the hypotheses made on components of the metric in our theorem on the energy estimate, we have to rewrite this equation with respect the system of coordinates (τ, x, v^A) used there. According to our previous calculations, equation (6.13) can be written as

$$
\eta^{\lambda\mu}\frac{\partial^2 \hat{f}}{\partial y^\lambda \partial y^\mu} + \Omega^{-\frac{n+3}{2}}H^{\lambda\mu}(f, \partial f)\frac{\partial^2 f}{\partial x^\lambda \partial x^\mu} = \Omega^{-\frac{n+3}{2}}F(f, \partial f) \,,
\tag{6.14}
$$

where
$$
\hat{f} = \Omega^{-\frac{n-1}{2}}f \,.
$$

So, let us express the second term of the above equation in terms of coordinates y^ν. We already know the identity:

$$
\frac{\partial^2 f}{\partial x^\lambda \partial x^\mu}\circ\phi^{-1} = \frac{\partial^2(f\circ\phi^{-1})}{\partial y^\alpha \partial y^\beta}A_\mu^\alpha A_\lambda^\beta + \frac{\partial(f\circ\phi^{-1})}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\lambda}\circ\phi^{-1} =: K_{\lambda\mu} + V_{\lambda\mu} \,,
\tag{6.15}
$$

with

$$
\frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\lambda}\circ\phi^{-1} = 2\Omega\delta_\mu^\alpha \eta_{\lambda\sigma}y^\sigma + 2\Omega\delta_\lambda^\alpha \eta_{\mu\tau}y^\tau + 2\Omega\eta_{\mu\lambda}y^\alpha + 8\eta_{\lambda\sigma}\eta_{\mu\theta}y^\sigma y^\alpha y^\theta
$$

and
$$
A_\mu^\alpha A_\lambda^\beta = \Omega^2\delta_\mu^\alpha \delta_\lambda^\beta + 4y_\lambda y_\mu y^\alpha y^\beta + 2\Omega(\delta_\mu^\alpha y_\lambda y^\beta + \delta_\lambda^\beta y_\mu y^\alpha) \,.
$$

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These identities lead to

$$H^{\lambda\mu}V_{\lambda\mu} = H^{\lambda\mu}\left\{2\Omega\delta^{\alpha}_{\mu}\eta_{\lambda\sigma}y^{\sigma} + 2\Omega\delta^{\alpha}_{\lambda}\eta_{\mu\theta}y^{\theta} + 2\Omega\eta_{\mu\lambda}y^{\alpha} + 8\eta_{\lambda\sigma}\eta_{\mu\theta}y^{\sigma}y^{\alpha}y^{\theta}\right\}\frac{\partial f}{\partial y^{\alpha}} \; .$$
$$(6.16)$$

Now we also know that

$$\frac{\partial f}{\partial y^{\alpha}} = \Omega^{\frac{n-3}{2}}\left\{\Omega\frac{\partial \hat{f}}{\partial y^{\alpha}} - (n-1)y_{\alpha}\hat{f}\right\} \; . \qquad (6.17)$$

This implies that [2]

$$H^{\lambda\mu}V_{\lambda\mu} = 2\Omega^{\frac{n-1}{2}}H^{\lambda\mu}\left\{(n-1)\left\{\Omega\eta_{\lambda\mu} + 2y_{\mu}y_{\lambda}\right\}\hat{f} + \left(2\Omega\delta^{\alpha}_{\mu}y_{\lambda} + \Omega\eta_{\lambda\mu}y^{\alpha} + 4y_{\mu}y_{\lambda}y^{\alpha}\right)\frac{\partial \hat{f}}{\partial y^{\alpha}}\right\} \; .$$
$$(6.18)$$

On the other hand we have

$$\frac{\partial^{2}(f\circ\phi^{-1})}{\partial y^{\alpha}\partial y^{\beta}} = \Omega^{\frac{n-5}{2}}\left\{\Omega^{2}\frac{\partial^{2}\hat{f}}{\partial y^{\alpha}\partial y^{\beta}} - (n-1)\Omega\left(y_{\beta}\frac{\partial \hat{f}}{\partial y^{\alpha}} + y_{\alpha}\frac{\partial \hat{f}}{\partial y^{\beta}}\right)\right.$$
$$\left. + (n-1)\left[(n-3)y_{\alpha}y_{\beta} - \Omega\eta_{\alpha\beta}\right]\hat{f}\right\} \; ,$$

which leads to the following expression of $H^{\lambda\mu}K_{\lambda\mu}$:

$$H^{\lambda\mu}K_{\lambda\mu} = \Omega^{\frac{n-5}{2}}H^{\lambda\mu}\left\{\Omega^{2}\delta^{\alpha}_{\mu}\delta^{\beta}_{\lambda} + 4y_{\mu}y_{\lambda}y^{\alpha}y^{\beta} + 2\Omega y^{\theta}\left(\eta_{\lambda\theta}\delta^{\alpha}_{\mu}y^{\beta} + \eta_{\mu\theta}\delta^{\beta}_{\lambda}y^{\alpha}\right)\right\}$$
$$\times \left\{\Omega^{2}\frac{\partial^{2}\hat{f}}{\partial y^{\alpha}\partial y^{\beta}} - (n-1)\Omega\left(y_{\beta}\frac{\partial \hat{f}}{\partial y^{\alpha}} + y_{\alpha}\frac{\partial \hat{f}}{\partial y^{\beta}}\right) + (n-1)\left[(n-3)y_{\alpha}y_{\beta} - \Omega\eta_{\alpha\beta}\right]\hat{f}\right\} \; ,$$

and after simplifications, we find that

$$H^{\lambda\mu}K_{\lambda\mu} = \Omega^{\frac{n-1}{2}}H^{\lambda\mu}\left\{\Omega^{2}\delta^{\alpha}_{\mu}\delta^{\beta}_{\lambda} + 4y_{\mu}y_{\lambda}y^{\alpha}y^{\beta} + 2\Omega(\delta^{\alpha}_{\mu}y_{\lambda}y^{\beta} + \delta^{\beta}_{\lambda}y_{\mu}y^{\alpha})\right\}\frac{\partial^{2}\hat{f}}{\partial y^{\alpha}\partial y^{\beta}}$$
$$+ (n-1)\Omega^{\frac{n-1}{2}}H^{\lambda\mu}\left\{2\left(2y_{\lambda}y_{\mu}y^{\alpha} + \Omega\delta^{\alpha}_{\lambda}y_{\mu}\right)\frac{\partial \hat{f}}{\partial y^{\alpha}} + \left[(n-3)y_{\mu}y_{\lambda} - \Omega\eta_{\lambda\mu}\right]\hat{f}\right\} \; .$$
$$(6.19)$$

---

[2]Note that in this equation, the term $y_{\mu}y_{\lambda}H^{\mu\lambda}$ is the one which has the the smallest multiplicative power of Ω.

With the expressions (6.18) and (6.19) and writing $H^{\lambda\mu} = \Omega^{\frac{n-1}{2}} \widehat{H}^{\lambda\mu}$, equation (6.14) reads after simplifications

$$\left\{\eta^{\alpha\beta} + \Omega^{\frac{n-5}{2}} \widehat{H}^{\lambda\mu} \left[\Omega^2 \delta^\alpha_\mu \delta^\beta_\lambda + 4 y_\mu y_\lambda y^\alpha y^\beta + 2\Omega(\delta^\alpha_\mu y_\lambda y^\beta + \delta^\beta_\lambda y_\mu y^\alpha)\right]\right\} \frac{\partial^2 \hat{f}}{\partial y^\alpha \partial y^\beta}$$

$$+ \quad 2\Omega^{\frac{n-5}{2}} \widehat{H}^{\lambda\mu} \left\{ \left\{2(n+1)y_\mu y_\lambda y^\alpha + (n+1)\Omega\delta^\alpha_\mu y_\lambda + \eta_{\lambda\mu}\Omega y^\alpha\right\} \frac{\partial \hat{f}}{\partial y^\alpha} \right.$$

$$\left. +(n-1)\left\{(n+1)y_\mu y_\lambda + \Omega\eta_{\lambda\mu}\right\} \hat{f} \right\}$$

$$= \quad \Omega^{-\frac{n+3}{2}} F\left(\Omega^{\frac{n-1}{2}} \hat{f}, \Omega^{(n-1)/2}\left\{-\Omega\frac{\partial}{\partial y^\nu} - 2y_\nu y^\alpha \frac{\partial}{\partial y^\alpha} - (n-1)y_\nu\right\}\hat{f}\right)$$

$$=: \quad \Omega^{-\frac{n+3}{2}} \widetilde{F}\left(\Omega^{\frac{n-1}{2}} \hat{f}, \Omega^{\frac{n-1}{2}} \frac{\partial \hat{f}}{\partial y^\nu}\right) . \tag{6.20}$$

We want to apply the energy estimates of Section 5.2.3 to the equation considered here. So for consistency of notation in that section, we write the above equation in the form (recall that $\Omega = x(\rho - \tau)$):

$$\Box_{\mathfrak{g}} u = \mathcal{F}(u, \partial u) , \tag{6.21}$$

with

$$u = \hat{f} , \tag{6.22}$$

$$\mathfrak{g}^{\alpha\beta} \quad = \quad \eta^{\alpha\beta} + \{x(\rho - \tau)\}^{\frac{n-5}{2}} \widehat{H}^{\lambda\mu} \times$$

$$\underbrace{\left\{\{x(\rho - \tau)\}^2 \delta^\alpha_\mu \delta^\beta_\lambda + 4 y_\mu y_\lambda y^\alpha y^\beta + 2\{x(\rho - \tau)\}(\delta^\alpha_\mu y_\lambda y^\beta + \delta^\beta_\lambda y_\mu y^\alpha)\right\}}_{:=\psi^{\alpha\beta}{}_{\mu\lambda}} ,$$

$$\tag{6.23}$$

(in order to reduce the typographical length of formulae we will sometimes write $\psi^{\alpha\beta}_{\mu\nu}$ for $\psi^{\alpha\beta}{}_{\mu\nu}$) and

$$\mathcal{F}\left(u, \frac{\partial u}{\partial y^\nu}\right) \quad = \quad \Omega^{-\frac{n+3}{2}} \widetilde{F}\left(\Omega^{\frac{n-1}{2}} u, \Omega^{\frac{n-1}{2}} \frac{\partial u}{\partial y^\nu}\right)$$

$$+ \left\{\Upsilon^\alpha - 2\Omega^{\frac{n-5}{2}} \widehat{H}^{\lambda\mu} \left\{2(n+1)y_\mu y_\lambda y^\alpha + (n+1)\Omega\delta^\alpha_\mu y_\lambda + \eta_{\lambda\mu}\Omega y^\alpha\right\}\right\} \frac{\partial u}{\partial y^\alpha}$$

$$-2(n-1)\Omega^{\frac{n-5}{2}} \widehat{H}^{\lambda\mu} \left\{(n+1)y_\mu y_\lambda + \Omega\eta_{\lambda\mu}\right\} u . \tag{6.24}$$

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So, we have to check that the metric $\mathfrak{g}$ defined by (6.23) and the harmonicity functions

$$\Upsilon^\mu = \frac{1}{\sqrt{|\det \mathfrak{g}|}} \partial_\nu \left\{ \sqrt{|\det \mathfrak{g}|} \mathfrak{g}^{\mu\nu} \right\} \tag{6.25}$$

satisfy the hypotheses of our theorem.

The tensor $\psi^{\alpha\beta}{}_{\mu\nu}$ defined in (6.23) has the property

$$\eta_{\alpha\beta} \psi^{\alpha\beta}{}_{\mu\nu} = \Omega^2 \eta_{\mu\nu} , \tag{6.26}$$

which implies that the contraction

$$\eta_{\alpha\beta} (\mathfrak{g}^{\alpha\beta} - \eta^{\alpha\beta}) = \Omega^{\frac{n-1}{2}} \operatorname{tr}_\eta \widehat{H}$$

gains two powers of Ω, as compared to a direct power-counting based on (6.23). Furthermore, the structure $y^\alpha y^\beta y_\mu y_\nu$ of the term without powers of Ω in $\psi^{\alpha\beta}{}_{\mu\nu}$ implies that any contraction of the form $\psi^{\alpha\beta}{}_{\mu\nu} \eta_{\alpha\rho} \psi^{\rho\sigma}{}_{\gamma\delta}$ acquires an overall multiplicative factor of Ω. So if we set

$$\delta\mathfrak{g}^\alpha{}_\beta := \mathfrak{g}^{\alpha\mu} \eta_{\mu\beta} - \delta^\alpha_\beta ,$$

it follows that for $k \geq 2$ we have

$$\left((\delta\mathfrak{g})^k \right)^\alpha{}_\beta := \delta\mathfrak{g}^\alpha{}_{\alpha_1} \delta\mathfrak{g}^{\alpha_1}{}_{\alpha_2} \cdots \delta\mathfrak{g}^{\alpha_{k-1}}{}_\beta = \Omega^{k-1} Q_k (\Omega^{\frac{n-5}{2}} \widehat{H}) ,$$

where we use the symbol Q_k to denote a smooth function (in this case, a polynomial) with a uniform zero of order k, and which may change from line to line. A similar analysis shows that, again for $k \geq 2$, the trace

$$p_k(\delta\mathfrak{g}) := \operatorname{tr}(\delta\mathfrak{g})^k = \delta\mathfrak{g}^\alpha{}_{\alpha_1} \delta\mathfrak{g}^{\alpha_1}{}_{\alpha_2} \cdots \delta\mathfrak{g}^{\alpha_{k-1}}{}_\alpha = \Omega^k Q_k (\Omega^{\frac{n-5}{2}} \widehat{H}) \tag{6.27}$$

(no summation over k) gains one more power of Ω.

Set

$$A^\alpha{}_\beta := \delta^\alpha_\beta + \delta\mathfrak{g}^\alpha{}_\beta . \tag{6.28}$$

Equation (6.27) implies

$$p_i(A) = \operatorname{tr}(I + \delta\mathfrak{g})^i = \sum_{j=0}^n C_i^j p_j(\delta\mathfrak{g}) = n + 1 + i \operatorname{tr}\delta\mathfrak{g} + \Omega^2 Q_2 (\Omega^{\frac{n-5}{2}} \widehat{H}) . \tag{6.29}$$

Let $W(\lambda)$ denote the characteristic polynomial of A,

$$W(\lambda) = \det(A - \lambda I) = \det A + w_1 \lambda + \ldots + w_n \lambda^n + (-\lambda)^{n+1} .$$

Then the coefficients w_i are homogeneous polynomials of order $n + 1 - i$ in the entries of $A = I + \delta\mathfrak{g}$, with $w_n = (-1)^n \mathrm{tr} A = (-1)^n(n + 1 + \mathrm{tr}\delta\mathfrak{g})$. It is a well known consequence of the Cayley-Hamilton theorem (see, e.g., [26, Theorem 1]) that both $\det A$ and the w_i's can be written as polynomials in the p_i's, and since each $p_i(A)$ has a factor Ω^2 in front of the Q_2 terms, we find that the w_i's take the form

$$w_i(A) = w_i(I) + \ell_i(\mathrm{tr}\delta\mathfrak{g}) + \Omega^2 Q_2(\Omega^{\frac{n-5}{2}}\widehat{H}) , \tag{6.30}$$

where $\ell_i(\mathrm{tr}\delta\mathfrak{g})$ is linear in $\mathrm{tr}\delta\mathfrak{g}$.

Now

$$\mathfrak{g}^{\alpha\beta} = \mathfrak{g}^{\alpha\sigma}\eta_{\sigma\rho}\eta^{\rho\beta} = \left(\delta^\alpha_\rho + \delta\mathfrak{g}^\alpha{}_\rho\right)\eta^{\rho\beta} = A^\alpha{}_\rho\eta^{\rho\beta} , \tag{6.31}$$

hence

$$\det \mathfrak{g}^\sharp = -\det(A) ,$$

which shows that

$$\det \mathfrak{g}^\sharp = -1 + \Omega^2 \left(-\Omega^{\frac{n-5}{2}}\mathrm{tr}_\eta \widehat{H} + Q_2(\Omega^{\frac{n-5}{2}}\widehat{H})\right) = -1 + \Omega^2 Q_1(\Omega^{\frac{n-5}{2}}\widehat{H}) . \tag{6.32}$$

From the Cayley-Hamilton theorem we have

$$A^{-1} = -\frac{1}{\det A}\left(w_1 I + \cdots + w_n A^{n-1} + (-1)^{n+1}A^n\right) ,$$

and we conclude that $\mathfrak{g}_{\alpha\beta} = (\eta^{-1}A^{-1})_{\alpha\beta}$ takes the form

$$
\begin{aligned}
\mathfrak{g}_{\alpha\beta} &= \frac{1}{1 + \Omega^2 Q_1(\Omega^{\frac{n-5}{2}})}\left(\eta_{\alpha\beta} - \Omega^{\frac{n-5}{2}}\widehat{H}^{\mu\nu}\psi_{\alpha\beta\mu\nu} + \Omega^2 Q_2(\Omega^{\frac{n-5}{2}}\widehat{H})\right) \\
&= \eta_{\alpha\beta} - \Omega^{\frac{n-5}{2}}\widehat{H}^{\mu\nu}y_\mu y_\nu y^\alpha y^\beta + \Omega Q_1(\Omega^{\frac{n-5}{2}}\widehat{H}) \\
&\quad + \Omega^2 Q_2(\Omega^{\frac{n-5}{2}}\widehat{H}) ,
\end{aligned} \tag{6.33}
$$

where the indices on $\psi_{\alpha\beta\mu\nu}$ have been lowered with the metric $\eta_{\alpha\beta}$.

6.1.3 On the components of the metric

Recall that, to obtain energy inequalities, our hypotheses on certain components of the metric were

$$\mathfrak{g}^{00} = -1 + x\mathfrak{h}^0; \quad \mathfrak{g}^{0\rho} = -x\mathfrak{h}^1; \quad \mathfrak{g}^{0A} + \mathfrak{g}^{\rho A} = -x\mathfrak{h}^A \quad \text{and} \quad \mathfrak{g}^{\rho\rho} = 1 + x\mathfrak{h} , \tag{6.34}$$

where the functions $\mathfrak{h}$, $\mathfrak{h}^0$, $\mathfrak{h}^A$ are bounded on bounded sets. Since (compare (6.4))

$$\mathfrak{g}^{0\rho} = \mathfrak{g}^{0i}\omega_i, \quad \mathfrak{g}^{0A} = \mathfrak{g}^{0i}\frac{\partial v^A}{\partial y^i}, \quad \mathfrak{g}^{\rho A} = \mathfrak{g}^{ij}\omega_j\frac{\partial v^A}{\partial y^i} \quad \text{and} \quad \mathfrak{g}^{\rho\rho} = \mathfrak{g}^{ij}\omega_i\omega_j ,$$

from (6.23) we have (note that $y^i\omega_i = \rho$, $\rho\omega_i\delta_\mu^i = y_\mu + \tau\delta_\mu^\tau$):

$$\mathfrak{h}^0 = x^{\frac{n-7}{2}}(\rho-\tau)^{\frac{n-5}{2}}\widehat{H}^{\lambda\mu}\left\{\{x(\rho-\tau)\}^2\delta_\mu^\tau\delta_\lambda^\tau + 4\tau^2 y_\mu y_\lambda + 4\tau\{x(\rho-\tau)\}\delta_\mu^\tau y_\lambda\right\} ,$$
$$(6.35)$$

$$\mathfrak{h}^1 = -x^{\frac{n-7}{2}}(\rho-\tau)^{\frac{n-5}{2}}\widehat{H}^{\lambda\mu}\left\{\{x(\rho-\tau)\}^2\delta_\mu^\tau\delta_\lambda^i\omega_i + 4\tau\rho y_\mu y_\lambda + 2\rho\{x(\rho-\tau)\}y_\lambda(\delta_\mu^0\rho + \tau\delta_\mu^i\omega_i)\right\} ,$$
$$(6.36)$$

$$\mathfrak{h} = x^{\frac{n-7}{2}}(\rho-\tau)^{\frac{n-5}{2}}\widehat{H}^{\lambda\mu}\left\{\{x(\rho-\tau)\}^2\delta_\mu^i\delta_\lambda^j\omega_i\omega_j + 4\rho^2 y_\mu y_\lambda + 4\{x(\rho-\tau)\}y_\lambda\rho\delta_\mu^i y_\lambda\omega_i\right\} ,$$
$$(6.37)$$

$$\mathfrak{h}^A = -x^{\frac{n-3}{2}}(\rho-\tau)^{\frac{n-3}{2}}\left\{(\rho-\tau)\left(\widehat{H}^{0i} + \omega_j\widehat{H}^{ij}\right) - 2y_\lambda\widehat{H}^{\lambda i}\right\}\frac{\partial v^A}{\partial y^i} . \quad (6.38)$$

We see that the components of the metric (6.23) have the right structure (6.34) if the space dimension n is greater then or equal to 7. We will see in Section 6.1.1 (see (6.12)) that this can be lowered to $n \geq 6$ using the harmonic coordinates condition.

We note the identities,

$$\eta^{ij}\omega_j\frac{\partial v^A}{\partial y^i} = \sum_{j=1}^{n}\omega_j\frac{\partial v^A}{\partial y^j} = \frac{\partial v^A}{\partial r} = 0 ,$$

which justify that $\mathfrak{g}^{0A} + \mathfrak{g}^{\rho A}$ has the right structure. In particular, for this component the condition $n \geq 4$ suffices to fulfill the structure condition.

We will also need

$$\mathfrak{g}^{\tau\tau} = -1 + O(x^{\frac{n-5}{2}}), \quad \mathfrak{g}^{\tau x} = 1 + O(x^{\frac{n-3}{2}}), \quad \mathfrak{g}^{xx} = O(x^{\frac{n-1}{2}}), \quad (6.39)$$
$$\mathfrak{g}^{xA} = O(x^{\frac{n-3}{2}}), \quad \mathfrak{g}^{AB} = \eta^{AB} + O(x^{\frac{n-5}{2}}) .$$

6.1.4 On the harmonicity functions

Now let us look at the harmonicity functions, defined as

$$\Upsilon^\mu := \frac{1}{\sqrt{|\det\mathfrak{g}|}}\partial_\nu\left\{\sqrt{|\det\mathfrak{g}|}\mathfrak{g}^{\mu\nu}\right\} .$$

Since our energy estimates have been established using the coordinate system (x, τ, v^A) as defined in (4.10), we need to calculate Υ^μ in that coordinate system. But so far we only have the expression of the metric in the

y^μ–coordinate system. To avoid confusion let us write $^{(2)}\Upsilon$ for Υ associated to the coordinates (τ, x, v^A) and $^{(1)}\Upsilon$ for that associated to the coordinates y^μ. To understand the behaviour of Υ under coordinate changes, it is useful to write the Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$ of the metric $\mathfrak{g}$ in the form

$$\Gamma^\alpha_{\beta\gamma} = \mathring{\Gamma}^\alpha_{\beta\gamma} + C^\alpha_{\beta\gamma} \,,$$

where the $\mathring{\Gamma}^\alpha_{\beta\gamma}$'s are the Christoffel symbols of the Minkowski metric η, and $C^\alpha_{\beta\gamma}$ is a tensor. Then, in the coordinate system y^μ we have

$$^{(1)}\Upsilon^\alpha = - \underbrace{\mathfrak{g}^{\beta\gamma}C^\alpha_{\beta\gamma}}_{=:C^\alpha} \,, \tag{6.40}$$

since the $\mathring{\Gamma}^\alpha_{\beta\gamma}$'s vanish in the y^μ–coordinates. Note that C^α as defined in (6.40) is a vector field, being the contraction of two tensors. In the coordinates (τ, x, v^A) we have

$$^{(2)}\Upsilon^\alpha = -\mathfrak{g}^{\beta\gamma}\left(\mathring{\Gamma}^\alpha_{\beta\gamma} + C^\alpha_{\beta\gamma}\right) = -\mathfrak{g}^{\beta\gamma}\mathring{\Gamma}^\alpha_{\beta\gamma} - C^\alpha \,. \tag{6.41}$$

Thus, to calculate $^{(2)}\Upsilon$ we need to vector-transform C^α to the (τ, x, v^A) coordinates, and calculate the missing term $\mathfrak{g}^{\beta\gamma}\mathring{\Gamma}^\alpha_{\beta\gamma}$ above. We start by calculating the vector field C^μ. We set

$$\mathfrak{g}^{\alpha\beta} =: \eta^{\alpha\beta} + \Omega^{\frac{n-5}{2}}K^{\alpha\beta} \,, \tag{6.42}$$

thus

$$K^{\alpha\beta} = \widehat{H}^{\mu\nu}\psi^{\alpha\beta}_{\mu\nu}$$

as in (6.23); we hope that the clash of notation with the completely different $K_{\alpha\beta}$ appearing in (6.15) will not confuse the reader.

From (6.32) we have (recall that Q means Q_2)

$$\left(\sqrt{|\det \mathfrak{g}|}\right)^{\mp 1} = 1 \pm \frac{1}{2}\Omega^{\frac{n-1}{2}}\mathrm{tr}_\eta(\widehat{H}) + \Omega^2 Q(\Omega^{\frac{n-5}{2}}\widehat{H}) \,.$$

Thus in the coordinate system y^μ,

$$\mathfrak{g}^{\mu\nu}\sqrt{|\det \mathfrak{g}|} = \eta^{\mu\nu}\left(1 - \frac{1}{2}\Omega^{\frac{n-1}{2}}\mathrm{tr}_\eta\widehat{H}\right) + \Omega^{\frac{n-5}{2}}K^{\mu\nu}$$
$$+ \Omega^2 Q^{\mu\nu}(\Omega^{\frac{n-5}{2}}\widehat{H}) \,, \tag{6.43}$$

$$\partial_\nu\left(\mathfrak{g}^{\mu\nu}\sqrt{|\det\mathfrak{g}|}\right) \;=\; \frac{1}{2}\Omega^{\frac{n-3}{2}}\left\{(n-1)y^\mu\mathrm{tr}_\eta\widehat{H}-\eta^{\mu\nu}\Omega\partial_\nu\mathrm{tr}_\eta\widehat{H}\right\}$$
$$+\Omega^{\frac{n-7}{2}}\left\{(5-n)y_\nu K^{\mu\nu}+\Omega\partial_\nu K^{\mu\nu}\right\}+\partial_\nu\left\{\Omega^2 Q^{\mu\nu}(\Omega^{\frac{n-5}{2}}\widehat{H})\right\}$$

and since

$$\partial_\nu\Omega K^{\mu\nu}\sim y_\nu K^{\mu\nu}=-\Omega y_\beta\widehat{H}^{\alpha\beta}\left\{\Omega\delta^\mu_\alpha+2y_\alpha y^\mu\right\}\;,\qquad(6.44)$$

and

$$\partial_\nu K^{\mu\nu}=\partial_\nu\widehat{H}^{\alpha\beta}\psi^{\mu\nu}_{\alpha\beta}+2(n+3)y_\beta\widehat{H}^{\alpha\beta}\left\{\Omega\delta^\mu_\alpha+2y_\alpha y^\mu\right\}+2\Omega y^\mu\mathrm{tr}_\eta\widehat{H}\;,\;\;(6.45)$$

we obtain

$$\partial_\nu\left(\mathfrak{g}^{\mu\nu}\sqrt{|\det\mathfrak{g}|}\right) \;=\; \frac{1}{2}\Omega^{\frac{n-3}{2}}\left\{(n+3)y^\mu\mathrm{tr}_\eta\widehat{H}-\eta^{\mu\nu}\Omega\partial_\nu\mathrm{tr}_\eta\widehat{H}\right\}$$
$$+\Omega^{\frac{n-5}{2}}\left\{\partial_\nu\widehat{H}^{\alpha\beta}\psi^{\mu\nu}_{\alpha\beta}+(3n+1)y_\beta\widehat{H}^{\alpha\beta}(\Omega\delta^\mu_\alpha+2y_\alpha y^\beta)\right\}$$
$$+\Omega^2 Q^\mu_2(\Omega^{\frac{n-5}{2}}\widehat{H})+\Omega^2 Q^{\mu\nu}_2(\Omega^{\frac{n-5}{2}}\widehat{H},\Omega^{\frac{n-5}{2}}\partial_\nu\widehat{H})\;.$$

Multiplying this last identity with $\left(\sqrt{|\det\mathfrak{g}|}\right)^{-1}$ we then obtain the following expression for the vector field C^μ:

$$C^\mu = {}^{(1)}\Upsilon^\mu \;=\; \frac{1}{2}\Omega^{\frac{n-3}{2}}\left\{(n+3)y^\mu\mathrm{tr}_\eta\widehat{H}-\eta^{\mu\nu}\Omega\partial_\nu\mathrm{tr}_\eta\widehat{H}\right\}$$
$$+\Omega^{\frac{n-5}{2}}\left\{\partial_\nu\widehat{H}^{\alpha\beta}\psi^{\mu\nu}_{\alpha\beta}+(3n+1)y_\beta\widehat{H}^{\alpha\beta}\left\{\Omega\delta^\mu_\alpha+2y_\alpha y^\mu\right\}\right\}$$
$$+\Omega^2 Q^\mu(\Omega^{\frac{n-5}{2}}\widehat{H})+\Omega^2 Q^{\mu\nu}(\Omega^{\frac{n-5}{2}}\widehat{H},\Omega^{\frac{n-5}{2}}\partial_\nu\widehat{H})\;.\quad(6.46)$$

Now writing the vector field C as

$$C=C^\mu\partial_\mu=:C^\tau\partial_\tau+C^x\partial_x+C^A\partial_A\;,$$

one is led to:

$$C^\tau=C^0\;,\quad C^\tau+C^x=-\omega_i(v)C^i\;,\quad C^A=\frac{\partial v^A}{\partial y^i}C^i\;.$$

In order to have all the harmonicity functions in the (τ,x,v^A)-coordinates, it remains to calculate the term $\mathfrak{g}^{\beta\gamma}\mathring{\Gamma}^\alpha_{\beta\gamma}$ of the formula (6.41). In these coordinates the Christoffell's symbol of the Minkowski metric $\mathring{\Gamma}^\alpha_{\beta\gamma}$ read:

$$\mathring{\Gamma}^\tau_{\alpha\beta} \;=\; 0,$$
$$\mathring{\Gamma}^x_{\tau\mu} \;=\; \mathring{\Gamma}^x_{x\mu}=0,\;\;\mathring{\Gamma}^x_{AB}=\rho\chi_{AB}$$
$$\mathring{\Gamma}^A_{\tau\tau} \;=\; \mathring{\Gamma}^A_{\tau x}=\mathring{\Gamma}^A_{xx}=0,\;\;\mathring{\Gamma}^A_{\tau B}=\mathring{\Gamma}^A_{xB}=-\frac{1}{\rho}\delta^A_B,\;\;\mathring{\Gamma}^A_{BC}=\gamma^A_{BC}\;,$$

where we have denoted the round metric on the sphere by χ, and its corresponding Christoffel symbols γ_{BC}^A. These identities lead to the following (see identity (6.42)):

$$\mathfrak{g}^{\beta\gamma}\mathring{\Gamma}_{\beta\gamma}^\tau \ = \ 0 \tag{6.47a}$$

$$\mathfrak{g}^{\beta\gamma}\mathring{\Gamma}_{\beta\gamma}^x \ = \ \rho\mathfrak{g}^{AB}\chi_{AB} = \frac{n-1}{\rho} + \rho\Omega^{\frac{n-5}{2}}\widehat{H}^{\mu\nu}\psi_{\mu\nu}^{AB}\chi_{AB} \tag{6.47b}$$

$$\mathfrak{g}^{\beta\gamma}\mathring{\Gamma}_{\beta\gamma}^A \ = \ -\frac{2}{\rho}(\mathfrak{g}^{\tau A} + \mathfrak{g}^{xA}) + \mathfrak{g}^{BC}\gamma_{BC}^A \tag{6.47c}$$

$$= \ \frac{1}{\rho^2}\mathring{C}^A + \Omega^{\frac{n-5}{2}}\widehat{H}^{\mu\nu}\left(2\psi_{\mu\nu}^{iA}\frac{\omega_i}{\rho} + \psi_{\mu\nu}^{BC}\gamma_{BC}^A\right) \ ; \tag{6.47d}$$

where $\mathring{C}^A = \chi^{BC}\gamma_{BC}^A$ is minus the harmonicity function on the unit sphere. Finally, we obtain that the harmonicity functions of the metric $\mathfrak{g}$ in the (τ, x, v^A)-coordinates read:

$$^{(2)}\Upsilon^\tau \ = \ -C^0 \tag{6.48a}$$

$$^{(2)}\Upsilon^\tau + {}^{(2)}\Upsilon^x \ = \ \omega_i(v)C^i - \frac{n-1}{\rho} - \rho\Omega^{\frac{n-5}{2}}\widehat{H}^{\mu\nu}\psi_{\mu\nu}^{AB} \tag{6.48b}$$

$$^{(2)}\Upsilon^A \ = \ -\frac{\partial v^A}{\partial y^i}C^i - \frac{1}{\rho^2}\mathring{C}^A$$

$$-\Omega^{\frac{n-5}{2}}\widehat{H}^{\mu\nu}\left(2\psi_{\mu\nu}^{iA}\frac{\omega_i}{\rho} + \psi_{\mu\nu}^{BC}\gamma_{BC}^A\right) \ . \tag{6.48c}$$

We revert now to the notation Υ for what was denoted by $^{(2)}\Upsilon$ above.

6.1.5 The source term $\mathcal{F}$

Recall that the source term in y^μcoordinates reads:

$$\mathcal{F}\left(u, \frac{\partial u}{\partial y^\nu}\right) \ = \ \Omega^{-\frac{n+3}{2}}\widetilde{F}\left(\Omega^{\frac{n-1}{2}}u, \Omega^{\frac{n-1}{2}}\frac{\partial u}{\partial y^\nu}\right)$$

$$+\left\{{}^{(1)}\Upsilon^\alpha - 2\Omega^{\frac{n-5}{2}}\widehat{H}^{\lambda\mu}\left\{2(n+1)y_\mu y_\lambda y^\alpha + (n+1)\Omega\delta_\mu^\alpha y_\lambda + \eta_{\lambda\mu}\Omega y^\alpha\right\}\right\}\frac{\partial u}{\partial y^\alpha}$$

$$-2(n-1)\Omega^{\frac{n-5}{2}}\widehat{H}^{\lambda\mu}\left\{(n+1)y_\mu y_\lambda + \Omega\eta_{\lambda\mu}\right\}u \ . \tag{6.49}$$

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From (6.46) we have

$$
{}^{(1)}\Upsilon^\alpha - 2\Omega^{\frac{n-5}{2}}\widehat{H}^{\lambda\mu}\left\{2(n+1)y_\mu y_\lambda y^\alpha + (n+1)\Omega\delta^\alpha_\mu y_\lambda + \eta_{\lambda\mu}\Omega y^\alpha\right\}
$$
$$
= \frac{1}{2}\Omega^{\frac{n-3}{2}}\left\{(n-1)y^\alpha \mathrm{tr}_\eta\widehat{H} - \eta^{\alpha\nu}\Omega\partial_\nu\mathrm{tr}_\eta\widehat{H}\right\}
$$
$$
+\Omega^{\frac{n-5}{2}}\left\{\psi^{\alpha\nu}_{\mu\lambda}\partial_\nu\widehat{H}^{\lambda\mu} + (n-1)y_\lambda\widehat{H}^{\mu\lambda}\left\{\Omega\delta^\alpha_\mu + 2y_\mu y^\alpha\right\}\right\}
$$
$$
+\Omega^2 Q^\alpha(\Omega^{\frac{n-5}{2}}\widehat{H},\Omega^{\frac{n-5}{2}}\widehat{H}) + \Omega^2 Q^{\alpha\beta}(\Omega^{\frac{n-5}{2}}\widehat{H},\Omega^{\frac{n-5}{2}}\partial_\beta\widehat{H})\ .
$$

This shows that the source term takes the following form:

$$
\mathcal{F}\left(u,\frac{\partial u}{\partial y^\nu}\right) = \Omega^{-\frac{n+3}{2}}\widetilde{F}\left(\Omega^{\frac{n-1}{2}}u,\Omega^{\frac{n-1}{2}}\frac{\partial u}{\partial y^\nu}\right)
$$
$$
-2(n-1)\Omega^{\frac{n-5}{2}}\widehat{H}^{\lambda\mu}\left\{(n+1)y_\mu y_\lambda + \Omega\eta_{\lambda\mu}\right\}u
$$
$$
+\frac{1}{2}\Omega^{\frac{n-3}{2}}\left\{(n-1)y^\alpha\mathrm{tr}_\eta\widehat{H} - \eta^{\alpha\nu}\Omega\partial_\nu\mathrm{tr}_\eta\widehat{H}\right\}\frac{\partial u}{\partial y^\alpha}
$$
$$
+\Omega^{\frac{n-5}{2}}\left\{\psi^{\alpha\nu}_{\mu\lambda}\partial_\nu\widehat{H}^{\lambda\mu} + (n-1)y_\lambda\widehat{H}^{\mu\lambda}\left\{\Omega\delta^\alpha_\mu + 2y_\mu y^\alpha\right\}\right\}\frac{\partial u}{\partial y^\alpha}
$$
$$
+\left\{\Omega^2 Q^\alpha(\Omega^{\frac{n-5}{2}}\widehat{H},\Omega^{\frac{n-5}{2}}\widehat{H}) + \Omega^2 Q^{\alpha\beta}(\Omega^{\frac{n-5}{2}}\widehat{H},\Omega^{\frac{n-5}{2}}\partial_\beta\widehat{H})\right\}\frac{\partial u}{\partial y^\alpha}\ .
$$
$$
(6.50)
$$

6.2 The Einstein-Maxwell case

6.2.1 Existence of a solution

The Einstein-Maxwell equations, in harmonic and Lorenz gauge, take the form (6.13) (see [4, 16, 18]) with the following replacements there:

$$
f = (\underbrace{g_{\mu\nu} - \eta_{\mu\nu}}_{:=\, h_{\mu\nu}}, A_\mu) \quad \text{and} \quad H^{\alpha\beta} = g^{\alpha\beta} - \eta^{\alpha\beta}\ . \tag{6.51}
$$

Recall that, if v is an arbitrary function, then

$$
\hat{v} = \Omega^{-\frac{n-1}{2}}v\ . \tag{6.52}
$$

Therefore, we have

$$
\hat{f} = (\hat{h}_{\mu\nu},\ \widehat{A}_\mu) := (\Omega^{-\frac{n-1}{2}}h_{\mu\nu},\ \Omega^{-\frac{n-1}{2}}A_\mu) \quad \text{and} \quad \widehat{H}^{\alpha\beta} = \Omega^{-\frac{n-1}{2}}H^{\alpha\beta}\ .
$$

For consistency of notation with Section 5.2.3 we set

$$
\hat{f} \equiv u\ .
$$

In this notation

$$\|\hat{h}_{\mu\nu}\|_{\mathscr{H}_k^\theta} \leq \|u\|_{\mathscr{H}_k^\theta} \ ,$$

and, since

$$\widehat{H}^{\alpha\beta} = -\eta^{\alpha\mu}\eta^{\beta\nu}\hat{h}_{\mu\nu} + \Omega^{-(n-1)/2}Q^{\alpha\beta}\left(\Omega^{(n-1)/2}\hat{h}_{\mu\nu}\right) \ ,$$

where $Q^{\alpha\beta}$ has a uniform zero of order two, from [13, Proposition A.2] we obtain that

$$
\begin{aligned}
\|\widehat{H}^{\alpha\beta}\|_{\mathscr{H}_k^\theta} &\leq \|\eta^{\alpha\mu}\eta^{\beta\nu}\hat{h}_{\mu\nu}\|_{\mathscr{H}_k^\theta} + \|\Omega^{-\frac{n-1}{2}}Q^{\alpha\beta}\left(\Omega^{\frac{n-1}{2}}\hat{h}_{\mu\nu}\right)\|_{\mathscr{H}_k^\theta} \\
&\leq C\left(\|\hat{h}_{\mu\nu}\|_{L^\infty}\right)\|u\|_{\mathscr{H}_k^{\theta-(n-1)/2}} \\
&\leq C\left(\|\hat{h}_{\mu\nu}\|_{L^\infty}\right)\|u\|_{\mathscr{H}_k^\theta} \ . \tag{6.53}
\end{aligned}
$$

We define the energy $E_{k,\lambda}^\alpha[u(\tau)]$ as in Equation (5.47) of Section (5.2.3), the metric being defined by (6.23). Recall (see Equation (5.49) of Section (5.2.3)) that this quantity controls the $\mathscr{H}_k^\alpha$-norms of $\partial\hat{f}$. Now,

$$\|\partial\widehat{H}^{\alpha\beta}\|_{\mathscr{H}_k^\theta}^2 \leq \|\partial(\eta^{\alpha\mu}\eta^{\beta\nu}\hat{h}_{\mu\nu})\|_{\mathscr{H}_k^\theta}^2 + \|\partial\left(\Omega^{-\frac{n-1}{2}}Q^{\alpha\beta}(\Omega^{(n-1)/2}\hat{h}_{\mu\nu})\right)\|_{\mathscr{H}_k^\theta}^2 \ .$$

Since

$$
\begin{aligned}
\partial\left(\Omega^{-\frac{n-1}{2}}Q^{\alpha\beta}(\Omega^{(n-1)/2}\hat{h}_{\mu\nu})\right) &= \Omega^{-\frac{n+1}{2}}Q^{\alpha\beta}(\Omega^{\frac{n-1}{2}}\hat{h}) + \Omega^{-\frac{n-1}{2}}Q^{\alpha\beta}(\Omega^{\frac{n-1}{2}}\hat{h}, \Omega^{\frac{n-1}{2}}\partial\hat{h}) \\
&= \Omega^{-\frac{n+1}{2}}Q^{\alpha\beta}(\Omega^{\frac{n-1}{2}}\hat{h}) + \Omega^{-\frac{n-1}{2}+\alpha}Q^{\alpha\beta}(\Omega^{\frac{n-1}{2}}(\hat{h}, x^{-\alpha}\partial\hat{h})),
\end{aligned}
$$

we have the estimate:

$$
\begin{aligned}
\|\partial\left(\Omega^{-\frac{n-1}{2}}Q^{\alpha\beta}(\Omega^{(n-1)/2}\hat{h}_{\mu\nu})\right)\|_{\mathscr{H}_k^\theta}^2 &\leq \|\Omega^{-\frac{n+1}{2}}Q^{\alpha\beta}(\Omega^{\frac{n-1}{2}}\hat{h})\|_{\mathscr{H}_k^\theta} \\
&\quad + \|\Omega^{-\frac{n-1}{2}+\alpha}Q^{\alpha\beta}(\Omega^{\frac{n-1}{2}}(\hat{h}, x^{-\alpha}\partial\hat{h}))\|_{\mathscr{H}_k^\theta} \\
&\leq C(\|\hat{h}\|_{L^\infty})\|\hat{h}\|_{\mathscr{H}_k^{\theta-(n-1)/2}} \\
&\quad + C(\|(\hat{h}, \ x^{-\alpha}\partial\hat{h})\|_{L^\infty})\|(\hat{h}, \ x^{-\alpha}\partial\hat{h})\|_{\mathscr{H}_k^{\theta-\alpha-(n-1)/2}} \\
&\leq C(\|(\hat{h}, \ x^{-\alpha}\partial\hat{h})\|_{L^\infty})\left(\|\hat{h}\|_{\mathscr{H}_k^\theta} + \|\hat{h}\|_{\mathscr{H}_k^{\theta-\alpha}} + \|\partial\hat{h}\|_{\mathscr{H}_k^\theta}\right) \\
&\leq C(\|(\hat{h}, \ x^{-\alpha}\partial\hat{h})\|_{L^\infty})\left(\|\hat{h}\|_{\mathscr{H}_k^{\theta-\alpha}} + \|\partial\hat{h}\|_{\mathscr{H}_k^\theta}\right) \ .
\end{aligned}
$$

Thus,

$$\|\partial\widehat{H}^{\alpha\beta}\|_{\mathscr{H}_k^\theta}^2 \leq C\left(\|(\hat{h}, \ x^{-\alpha}\partial\hat{h})\|_{L^\infty}\right)\left(\|u\|_{\mathscr{H}_k^{\theta-\alpha}} + \|\partial u\|_{\mathscr{H}_k^\theta}\right) \ .$$

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To continue, we suppose that at $x = x_1 > 0$ the maximal globally hyperbolic development of the data exists for $\tau \in [\tau_0, \tau_1]$, with

$$M_1 := \|\hat{f}|_{\{x=x_1\}}\|_{L^\infty} < \infty .$$

We define (compare (5.51))

$$
\begin{aligned}
\hat{M}(\tau) \quad := \quad & \|\mathcal{F}\|^2_{\mathscr{B}^\alpha_0(\mathbf{H}_\tau)} + \|(\mathfrak{g}, (\partial_\tau - \partial_x)\mathfrak{g}^\sharp)\|^2_{L^\infty(\mathbf{H}_\tau)} + \|(\mathfrak{g}^\sharp, \mathfrak{h}^\sharp, \Upsilon)\|^2_{\mathscr{C}^0_{\{x=0\},1}(\mathbf{H}_\tau)} \\
& + \|((\partial_\tau - \partial_x)\hat{f}, \partial_x\hat{f}, \partial_A\hat{f})\|^2_{\mathscr{B}^\alpha_1(\mathbf{H}_\tau)} + \|\hat{f}(\tau)|_{\{x=x_1\}}\|_{L^\infty} , \quad\quad (6.54)
\end{aligned}
$$

with the functions $\mathfrak{g}^\sharp$, $\mathfrak{h}^\sharp$, $\Upsilon^\mu \equiv {}^{(2)}\Upsilon$ and $\mathcal{F}$ defined by equations (6.23), (6.35)-(6.38), (6.48) and (6.50).

For any positive function $N(\tau)$ we set

$$\underline{N(\tau)} := \sup_{s \in [\tau_0, \tau]} N(s) . \qquad (6.55)$$

We then have the following:

PROPOSITION 6.1 *Let $k \in \mathbb{N}$, $\alpha \in (-1, -1/2]$. Consider the Einstein-Maxwell equations (6.13) in space-time dimension $1+n \geq 7$ if $\alpha = -\frac{1}{2}$, and $1+n \geq 8$ otherwise. Let f be defined in (6.51), suppose that $t_0 > 0$ and assume that the initial data, given on the hyperboloid*

$$\mathscr{S}_0 = \left\{ (x^\mu) : x^0 - t_0 = \sqrt{t_0^2 + |\vec{x}|^2} \right\} \qquad (6.56)$$

in Minkowski space-time, are such that:

$$\hat{f}\big|_{\phi(\mathscr{S}_0)} \in \left(\mathscr{H}^\alpha_{k+1} \cap L^\infty\right)(\phi(\mathscr{S}_0)) , \quad \text{and} \quad \left((\partial_\tau - \partial_x)\hat{f}, \partial_x\hat{f}, \partial_A\hat{f}\right)\big|_{\phi(\mathscr{S}_0)} \in \mathscr{H}^\alpha_k(\phi(\mathscr{S}_0)) . \tag{6.57}$$

There exists functions $\hat{C}_3(n, k, \epsilon_0, C_0, \alpha, \hat{M})$ and $\hat{C}_4(n, k, \epsilon_0, C_0, \alpha, \hat{M})$, monotonously increasing in $\hat{M}$, which we write as $\hat{C}_3(\hat{M})$ and $\hat{C}_4(\hat{M})$, such that the energy of the system as defined in (5.47), Section 5.2.3 satisfies the inequality

$$
\begin{aligned}
\|\hat{f}(\tau)\|^2_{L^\infty} + E^\alpha_k[\hat{f}(\tau)] \quad \leq \quad & 2\Big\{ M_1^2 + E^\alpha_k[\hat{f}(\tau_0)] \\
& + \int_{\tau_0}^\tau \hat{C}_3(\hat{M}(s)) E^\alpha_k[\hat{f}(s)] ds \Big\} , \qquad (6.58)
\end{aligned}
$$

where $\tau_0 = -\frac{1}{2t_0}$. Furthermore, for $n + 1 \geq 7$ and $\alpha = -1/2$ one has

$$
\begin{aligned}
\underline{\|\hat{f}(\tau)\|^2_{L^\infty}} + \underline{E^\alpha_k[\hat{f}(\tau)]} \quad \leq \quad & 2\Big\{ M_1^2 + E^\alpha_k[\hat{f}(\tau_0)] + \|x^{(n-7)/2}\hat{H}^{\mu\nu}y_\mu y_\nu(\tau_0)\|^2_{\mathscr{G}^0_k} \\
& + \int_{\tau_0}^\tau \hat{C}_4(\underline{\hat{M}(s)})\underline{E^\alpha_k[\hat{f}(s)]} ds \Big\} . \qquad (6.59)
\end{aligned}
$$

REMARK 6.2 For $n \geq 7$, a prefactor $\Omega^{\frac{n-7}{2}}$ in the fourth line[3] of the nonlinear term in (6.50) still leads to the estimates here. This remark is important for the estimation of the time derivatives in Section 6.2.3 below.

PROOF: For all $0 < x < x_1$ the trivial identity

$$\hat{f}(\tau, x) = \hat{f}(\tau, x_1) - \int_x^{x_1} \partial_x \hat{f}(\tau, s) ds$$

leads to the estimate (recall that $\alpha > -1$)

$$\begin{aligned}
\|\hat{f}(\tau)\|_{L^\infty} &\leq M_1 + \int_x^{x_1} \|\partial_x \hat{f}(\tau)\|_{\mathscr{C}^\alpha_{\{x=0\},0}} s^\alpha ds \\
&\leq M_1 + \|\partial_x \hat{f}(\tau)\|_{\mathscr{G}^\alpha_k} .
\end{aligned}$$

From this one easily concludes

$$\|\hat{f}(\tau)\|_{\mathscr{G}^0_k} \leq C\big(M_1 + \|(\partial_x \hat{f}, \partial_A \hat{f})(\tau)\|_{\mathscr{G}^\alpha_{k-1}}\big) . \tag{6.60}$$

Now we apply Proposition 5.10 of Section 5.2.3. To obtain (6.58) we will show first that, in the Einstein-Maxwell case, the $\mathscr{H}^\alpha_k$-norm of the source term, the $\mathscr{G}^0_k$-norms of $\mathfrak{g}^\sharp$, $\mathfrak{h}^\sharp$ and Υ^μ are controlled by the energy. Let us start with the $\mathscr{G}^0_k$-norm of $\mathfrak{g}^\sharp$. From the expression of $\mathfrak{g}$ given by (6.23) and the estimate (6.60), if $n \geq 5$ then

$$\begin{aligned}
\|\mathfrak{g}^\sharp(\tau)\|^2_{\mathscr{G}^0_k} &\leq C\left(M_1 + \|\partial \widehat{H}\|^2_{\mathscr{G}^\alpha_k}\right) \\
&\leq C\left(M_1 + E^\alpha_{k,\lambda}[u(\tau)]\right) . \tag{6.61}
\end{aligned}$$

The same holds for $\mathfrak{h}^\sharp$ but with the constraint that the space dimension n is larger than or equal to 7. We will return later to the question how to improve on the dimension on this term when $\alpha = -1/2$.

To estimate the harmonicity functions $^{(2)}\Upsilon$ given by (6.48), we start by estimating the functions C^μ. We decompose $C^\mu = C^\mu_1 + C^\mu_2 + C^\mu_3$, each corresponding to a line in (6.46). The first and second terms are estimated as we did for $\mathfrak{g}^\sharp$ and $\mathfrak{h}^\sharp$:

$$\begin{aligned}
\|C^\mu_1\|^2_{\mathscr{G}^0_k} &\leq C(\|x^{\frac{n-3}{2}} \widehat{H}\|^2_{\mathscr{G}^0_k} + \|x^{\frac{n-1}{2}} \partial \widehat{H}\|^2_{\mathscr{G}^0}) \\
&\leq C\big(M_1 + E^\alpha_{k,\lambda}[u(\tau)]\big) \quad \text{for} \quad n \geq 3 , \tag{6.62}
\end{aligned}$$

---

[3]The fall-off of the component of this term with the lowest power of Ω can be improved using the gauge condition.

and

$$
\begin{aligned}
\|C_2^\mu\|_{\mathscr{G}_k^0}^2 &\leq C\big(\|x^{\frac{n-5}{2}}\widehat{H}\|_{\mathscr{G}_k^0}^2 + \|x^{\frac{n-5}{2}}\partial\widehat{H}\|_{\mathscr{G}_k^0}^2\big) \\
&\leq C\big(M_1 + E_{k,\lambda}^\alpha[u(\tau)]\big) \quad \text{for} \quad n \geq 5 - 2\alpha\,. \qquad (6.63)
\end{aligned}
$$

To estimate C_3^μ we recall that its components have a uniform zero of order two in $\widehat{H}$ and $(\widehat{H}, x^{-\alpha}\partial\widehat{H})$ respectively, with the second term linear in $\partial\widehat{H}$, thus we can apply inequality (A.31) of [13] on the $\mathscr{G}$-norm with $\ell = 2$, $\beta = \frac{n-5}{2}$. We obtain:

$$
\begin{aligned}
\|C_3^\mu\|_{\mathscr{G}_k^0}^2 &\leq \|Q^\mu(\Omega^{\frac{n-5}{2}}\widehat{H})\|_{\mathscr{G}_k^{-2}}^2 + \|Q^{\mu\nu}(\Omega^{\frac{n-5}{2}}(\widehat{H}, x^{-\alpha}\partial_\nu\widehat{H}))\|_{\mathscr{G}_k^{-2-\alpha}}^2 \\
&\leq C(\|(\widehat{H}\|_{L^\infty})\|\widehat{H}\|_{\mathscr{G}_k^{3-n}}^2 + C(\|(\widehat{H}, x^{-\alpha}\partial\widehat{H})\|_{L^\infty})\|(\widehat{H}, x^{-\alpha}\partial\widehat{H})\|_{\mathscr{G}_k^{3-\alpha-n}}^2 \\
&\leq C\left(\|\widehat{H}, x^{-\alpha}\partial\widehat{H})\|_{L^\infty}\right)\left(\|\widehat{H}\|_{\mathscr{G}_k^0}^2 + \|\partial\widehat{H}\|_{\mathscr{G}_k^\alpha}^2\right) \quad \text{for} \quad n \geq 3 - \alpha\,.
\end{aligned}
$$

Thus we have:

$$
\|C_3^\mu\|_{\mathscr{G}_k^0}^2 \leq C\left(\|\widehat{H}, x^{-\alpha}\partial\widehat{H})\|_{L^\infty}\right)\left(\|u\|_{L^\infty}^2 + E_{k,\lambda}^\alpha[u(\tau)]\right) \quad \text{for} \quad n \geq 4\,.
$$
$$(6.64)$$

Note that the function $C\left(\|\widehat{H}, x^{-\alpha}\partial\widehat{H})\|_{L^\infty}\right)$ will give a contribution to the function $C_2(M(s))$ of (6.58). The remaining terms of $^{(2)}\Upsilon$ as given by (6.47) are estimated in a similar way. They are controlled by

$$
C\left(1 + E_{k,\lambda}^\alpha[u(\tau)]\right) \qquad \text{for} \quad n \geq 5\,. \qquad (6.65)
$$

We continue by writing the source term $\mathcal{F}$ (see (6.50)) as a sum of terms, each of the following form

$$
x^{p_i}\mathcal{F}_i\left(\,.\,, x^{q_i}(\hat{f},\, x^{-\alpha}\partial\hat{f})\right)\,. \qquad (6.66)
$$

Note that all terms are polynomial in ∂f, at most quadratic in ∂f. For instance, the first term $\tilde{F}$ arises from products of the Christoffels in the Ricci tensor, and from the products of the derivatives ∂A of the vector potential A in the energy-momentum tensor. We then write, for example, in the x^μ coordinates,

$$
\Gamma^2 \sim (g^\sharp\partial g)^2 \sim F(g^\sharp)\partial g\partial g = x^{2\alpha}F(g^\sharp)(x^{-\alpha}\partial g)(x^{-\alpha}\partial g)\,;
$$

we then express this in term of $h_{\mu\nu}$, transform the whole expression to the y^μ–coordinates, and finally reexpress $h_{\mu\nu}$ in term of $\widehat{H}_{\mu\nu}$. This formula

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$\mathcal{F}_i$	p_i	q_i	ℓ_i	constraint	$n \geq$
$\mathcal{F}_1$	$-\frac{n+3}{2}$	$\frac{n-1}{2}+\alpha$	2	$n > 5 - 2\alpha$	7[6]
$\mathcal{F}_2$	$-\frac{n-5}{2}$	$\frac{n-5}{2}$	2	$n > 5$	6
$\mathcal{F}_3$	$\frac{n-3}{2}+\alpha$	0	2	$n > 3$	4
$\mathcal{F}_4$	$\frac{n-5}{2}+2\alpha$	0	2	$n > 5 - 2\alpha$	7[6]
$\mathcal{F}_5$	$2-\frac{n-5}{2}+\alpha$	$\frac{n-5}{2}$	3	$n > 3$	4
	$2-\frac{n-5}{2}+2\alpha$	$\frac{n-5}{2}$	3	$n > 3 - \alpha$	4

Table 6.1: Restrictions on the dimension from the source terms.

shows that the Γ^2 in the Einstein equations have a uniform zero of order two in $(\hat{f}, x^{-\alpha}\hat{f})$. A similar analysis applies to the contribution of the Maxwell fields to the Einstein-Maxwell equations.

We use the following estimate to show that the $\mathscr{H}_k^\alpha$-norm of $\mathcal{F}$ is controlled by the energy of the system: Suppose that $\mathcal{F}_i$ has a uniform zero of order ℓ_i in $(u,\ x^{-\alpha}\partial u)$, then applying to this function the second part of lemma A.2 of [13], for

$$p_i + \ell_i q_i > \alpha . \tag{6.67}$$

We choose $\epsilon > 0$ so that $p_i + \ell_i q_i > \alpha + \epsilon$, and write

$$
\begin{aligned}
&\|x^{p_i}\mathcal{F}_i\left(\ .\ , x^{q_i}(u,\ x^{-\alpha}\partial u)\right)\|^2_{\mathscr{H}_k^\alpha}\\
&= \ \|\mathcal{F}_i\left(\ .\ , x^{q_i}(u, x^{-\alpha}\partial u)\right)\|^2_{\mathscr{H}_k^{\alpha-p_i}}\\
&\leq \ C\big(\|(u,\ x^{-\alpha}\partial u)\|_{L^\infty}\big)\|(u,\ x^{-\alpha}\partial u)\|^2_{\mathscr{H}_k^{\alpha-p_i-\ell_i q_i}}\\
&\leq \ C\big(\|(u,\ x^{-\alpha}\partial u)\|_{L^\infty}\big)\left(\|u\|^2_{\mathscr{H}_k^{-\epsilon}} + \|x^{-\alpha}\partial u\|^2_{\mathscr{H}_k^{-\epsilon}}\right)\\
&\leq \ C\big(\|(u,\ x^{-\alpha}\partial u)\|_{L^\infty}\big)\left(\|u\|^2_{L^\infty} + E_{k,\lambda}^\alpha[u(\tau)]\right) .
\end{aligned}
$$

The analysis of the nonlinear terms (6.50) along those lines gives the following table: Here the $\mathcal{F}_i$'s, $i = 1,\ldots,4$, correspond to the i-th line of (6.50), while the two rows for $\mathcal{F}_5$ correspond to the two respective terms in the last line of (6.50). In the last column the number in square bracket is obtained by estimating below the non-linearity in a more efficient way.

It turns out that the threshold on the space dimension n can be lowered to $n = 6$ for the components $\mathcal{F}_1$ and $\mathcal{F}_4$ of the source term $\mathcal{F}$. The quadratic terms in those expressions with the lowest powers of Ω are of the form $\Omega^{\frac{n-1}{2}}G(\Omega^{\frac{n-1}{2}}\hat{f})\partial\hat{f}\partial\hat{f}$ for $\mathcal{F}_1$ and $\Omega^{\frac{n-5}{2}}\widehat{H}\partial u$ and $\Omega^{\frac{n-5}{2}}\partial\widehat{H}\partial u$ for $\mathcal{F}_4$. One can

estimate the $\mathscr{H}_k^\alpha$-norm of $\Omega^{\frac{n-5}{2}}\widehat{H}\partial u$ using instead (5.100):

$$
\begin{aligned}
\|\Omega^{\frac{n-5}{2}}\widehat{H}\partial u\|^2_{\mathscr{H}_k^\alpha} \quad &\leq \quad \|\widehat{H}\partial u\|^2_{\mathscr{H}_k^{\alpha-\frac{n-5}{2}}} \\[2mm]
&\leq \quad C\left(\|\widehat{H}\|^2_{\mathscr{C}_0^0}\|\partial u\|^2_{\mathscr{H}_k^{\alpha-\frac{n-5}{2}}} + \|x^{\frac{n-5}{2}}\widehat{H}\|^2_{\mathscr{G}_k^0}\|\partial u\|^2_{\mathscr{B}_0^\alpha} \right) \\[2mm]
&\leq \quad C(\|u\|^2_{L^\infty} + \|\partial u\|^2_{\mathscr{B}_0^\alpha})\left(\|u\|^2_{\mathscr{G}_k^0} + \|\partial u\|^2_{\mathscr{H}_k^\alpha} \right) \quad \text{if} \quad n \geq 5 \\[2mm]
&\underbrace{\leq}_{\text{see }(6.60)} \quad C(\|u\|^2_{L^\infty} + \|\partial u\|^2_{\mathscr{B}_0^\alpha})\left(1 + \|\partial u\|^2_{\mathscr{H}_k^\alpha} \right) \\[2mm]
&\leq \quad C(\|u\|^2_{L^\infty} + \|\partial u\|^2_{\mathscr{H}_k^\alpha})\left(1 + \|\partial u\|^2_{\mathscr{H}_k^\alpha} \right) , \quad\quad\quad (6.68)
\end{aligned}
$$

for $k > n/2$. Next,

$$
\begin{aligned}
\|\Omega^{\frac{n-5}{2}}\partial\widehat{H}\partial u\|^2_{\mathscr{H}_k^\alpha} \quad &\leq \quad \|\partial\widehat{H}\partial u\|^2_{\mathscr{H}_k^{\alpha-\frac{n-5}{2}}} \\[2mm]
&\leq \quad C\left(\|\partial\widehat{H}\|^2_{\mathscr{C}_0^\alpha}\|\partial u\|^2_{\mathscr{H}_k^{-\frac{n-5}{2}}} + \|\partial\widehat{H}\|^2_{\mathscr{H}_k^\alpha}\|\partial u\|^2_{\mathscr{C}_0^{-\frac{n-5}{2}}} \right) \\[2mm]
&\leq \quad C\|\partial u\|^2_{\mathscr{C}_0^\alpha}\|\partial u\|^2_{\mathscr{H}_k^\alpha} \quad \text{if} \quad -\frac{n-5}{2} - \alpha \leq 0 \quad \text{i.e.} \quad n \geq 5 - 2\alpha \\[2mm]
&\leq \quad C\|\partial\widehat{H}\|^2_{\mathscr{C}_0^\alpha} E^\alpha_{\lambda,k}[u(\tau)] ,
\end{aligned}
$$

and so the last inequality will be true provided that

$$
\left\{ \begin{array}{ll} n \geq 6 & \text{if} \quad \alpha = -\frac{1}{2} \\ n \geq 7 & \text{if} -1 < \alpha < -\frac{1}{2} \end{array} \right. \quad .
$$

A similar calculation applies to $\mathcal{F}_1$.

These estimates and the table show that

$$
\|\mathcal{F}(u, \partial u)\|^2_{\mathscr{H}_k^\alpha} \leq C(\|u\|_{L^\infty}, \|\partial u\|_{\mathscr{C}_0^\alpha})\left(1 + E^\alpha_{k,\lambda}[u(\tau)] \right) \quad\quad\quad (6.69)
$$

for

$$
\left\{ \begin{array}{ll} n \geq 6 & \text{if} \quad \alpha = -\frac{1}{2} \\ n \geq 7 & \text{if} -1 < \alpha < -\frac{1}{2} \end{array} \right. \quad .
$$

Inserting inequalities (6.61)-(6.63) and (6.64)-(6.69) in (5.52) of Section 5.2.3 gives (6.58).

Now, at several places of the calculations above the term

$$
\psi := y_\alpha y_\beta \hat{H}^{\alpha\beta}
$$

is the one that occurs with the lowest power of Ω. It follows from the wave-coordinates conditions that this term solves equation (6.12), which can be written in the form

$$-y^\alpha \partial_\alpha \psi + \frac{n-5}{2}\psi = \zeta \ , \qquad (6.70)$$

where

$$
\begin{aligned}
\zeta \ :=& \ \Omega\Big(\frac{n-1}{2}\mathrm{tr}_\eta(\widehat{H}) + y_\nu \frac{\partial}{\partial y^\mu}\big(\widehat{H}^{\mu\nu} + \frac{1}{2}\eta^{\mu\nu}\mathrm{tr}_\eta(\widehat{H})\big)\Big) \\
&+ \Omega^{-\frac{n-1}{2}}Q(\Omega^{\frac{n-1}{2}}\widehat{H}, \Omega^{\frac{n-1}{2}}\widehat{H}) \\
&+ \Omega^{-\frac{n-1}{2}}Q(\Omega^{\frac{n-1}{2}}\widehat{H}, \Omega^{\frac{n+1}{2}}\partial\widehat{H}) \\
&+ \Omega^{-\frac{n-1}{2}}Q(\Omega^{\frac{n-1}{2}}\widehat{H}, \Omega^{\frac{n-1}{2}}y^\alpha\frac{\partial}{\partial y^\alpha}\widehat{H}) \\
=:& \ \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 \ , \qquad (6.71)
\end{aligned}
$$

where ζ_i corresponds to the i-the line. The point is that all terms in ζ contain effectively multiplicative powers of Ω.

Solutions of (6.70) take the form, for $\tau_0 \le \tau \le \tau_1 < 0$,

$$\psi(\tau, x) = (-\tau)^{-(n-5)/2}\left(\int_{\tau_0}^\tau (-s)^{(n-7)/2}\zeta\left(s, \frac{sx}{\tau}\right)ds + (-\tau_0)^{(n-5)/2}\psi\left(\tau_0, \frac{x\tau_0}{\tau}\right)\right) \ . \qquad (6.72)$$

This gives immediately, for any γ,

$$\|\psi(\tau)\|_{\mathscr{G}_k^\gamma} \le \|\psi(\tau_0)\|_{\mathscr{G}_k^\gamma} + C(\tau_0, \tau_1)\int_{\tau_0}^\tau \|\zeta(s)\|_{\mathscr{G}_k^\gamma}ds \ , \qquad (6.73)$$

similarly for $\mathscr{H}^\gamma$- or $\mathscr{C}^\gamma$-norms. In the notation of (6.55) one thus finds

$$
\begin{aligned}
\underline{\|\psi(\tau)\|_{\mathscr{G}_k^\gamma}} \ \le& \ \|\psi(\tau_0)\|_{\mathscr{G}_k^\gamma} + C(\tau_0, \tau_1)\int_{\tau_0}^\tau \|\zeta(s)\|_{\mathscr{G}_k^\gamma}ds \\
\le& \ \|\psi(\tau_0)\|_{\mathscr{G}_k^\gamma} + C(\tau_0, \tau_1)(\tau_1 - \tau_0)\underline{\|\zeta(\tau)\|_{\mathscr{G}_k^\gamma}} \ .
\end{aligned}
$$

Using this to estimate $\mathfrak{h}^0$ we obtain

$$
\begin{aligned}
\underline{\|\mathfrak{h}^0(\tau)\|_{\mathscr{G}_k^0}} \ \le& \ C\big(\|x^{(n-7)/2}\psi(\tau)\|_{\mathscr{G}_k^0} + \|x^{(n-5)/2}\widehat{H}(\tau)\|_{\mathscr{G}_k^0} \\
\le& \ C\big(\|x^{(n-7)/2}\psi(\tau_0)\|_{\mathscr{G}_k^0} + \underline{\|x^{(n-7)/2}\zeta(\tau)\|_{\mathscr{G}_k^0}} + \underline{\|x^{(n-5)/2}\widehat{H}(\tau)\|_{\mathscr{G}_k^0}}\big) \ .
\end{aligned}
$$

We have, for example,

$$\underline{\|x^{(n-7)/2}\zeta_1(\tau)\|_{\mathscr{G}_k^0}} \ \le \ C\left(\underline{\|x^{(n-5)/2}\widehat{H}(\tau)\|_{\mathscr{G}_k^0}} + \underline{\|x^{(n-5)/2}\partial\widehat{H}(\tau)\|_{\mathscr{G}_k^0}}\right) \ ,$$

which, for $n - 5 \geq -2\alpha$, can be controlled by $\|\widehat{H}(\tau)\|_{L^\infty}$ and $E_k^\alpha[u(\tau)]$ in view of (6.60). This requires $n \geq 6$ if $\alpha = -1/2$, or $n \geq 7$ if $\alpha \in (-1, -1/2)$. An estimation of the remaining ζ_i's along the lines of those already done above presents no difficulties.

The functions $\mathfrak{h}^1$ and $\mathfrak{h}$ have the same structure and so the same estimate applies; the function $\mathfrak{h}^A$ has a higher multiplicative power of Ω so that the original straightforward estimate applies.

The final inequality (6.59) follows immediately from this and from an obvious version of the estimate (6.58) for the remaining terms in the equation.

We finish this proof by noting that the above treatment of $y_\alpha y_\beta \hat{H}^{\alpha\beta}$ can be used to improve the threshold on dimension for some of the entries of Table 6.1; this will, however, not improve the threshold on n of the theorem.
$\square$

We are now ready to prove existence of solutions in weighted Sobolev spaces. For $s > 0$ consider the family of hyperboloids:

$$\mathscr{S}_s = \left\{ (x^\mu) : x^0 - s = \sqrt{s^2 + |\vec{x}|^2} \right\} . \tag{6.74}$$

Let ϕ be defined in (3.2). We have the following

THEOREM 6.3 (Propagation of weighted Sobolev regularity) *Suppose that $k > \left[\frac{n}{2}\right] + 1$, with $n = 6$ and $\alpha = -1/2$, or $n \geq 7$ with $\alpha \in (-1, 1/2]$, and let $t_0 > 0$. Suppose that*

$$\hat{f}\big|_{\phi(\mathscr{S}_0)} \in \left(\mathscr{H}_{k+1}^\alpha \cap L^\infty \right) (\phi(\mathscr{S}_0)) , \qquad \left(\partial_\tau \hat{f}, \partial_x \hat{f}, \partial_A \hat{f} \right) \big|_{\phi(\mathscr{S}_0)} \in \mathscr{H}_k^\alpha(\phi(\mathscr{S}_0)) , \tag{6.75}$$

where f and $\hat{f}$ are defined by (6.51)-(6.52). In the case $\alpha = -1/2$ and $n = 6$ assume moreover that

$$x^{-1/2} y_\alpha y_\beta \widehat{H}^{\alpha\beta}\big|_{\phi(\mathscr{S}_0)} \in \mathscr{G}_k^0 . \tag{6.76}$$

Then there exists $t_* > t_0$ and a solution of (6.13) defined on $\underset{s \in [t_0,\, t_*]}{\cup} \mathscr{S}_s$ such that, $\forall \tau \in [-\frac{1}{2t_0}, -\frac{1}{2t_*}] =: [\tau_0, \tau_*]$ we have:

$$\hat{f} \in L^\infty \left([\tau_0, \tau_*], \ \mathscr{H}_k^\alpha(\mathbf{H}_\tau) \cap L^\infty(\mathbf{H}_\tau) \right) , \tag{6.77}$$

$$\left(\partial_\tau \hat{f}, \ \partial_x \hat{f}, \ \partial_A \hat{f} \right) \in L^\infty \left([\tau_0, \tau_*], \ \mathscr{H}_k^\alpha(\mathbf{H}_\tau) \right) . \tag{6.78}$$

Moreover, any solution for which $\hat{M}(\tau)$, as defined in (6.54), is bounded on $[\tau_0, \tau_1]$ satisfies (6.77)-(6.78) with $\tau_* = \tau_1$.

REMARK 6.4 Using the weighted Sobolev embedding theorem we conclude

$$\hat{f}(\tau) \in \left(\mathscr{C}^\alpha_{k-\left[\frac{n}{2}\right]-1} \cap L^\infty \right) (\mathbf{H}_\tau) , \qquad (6.79)$$

$$\left(\partial_\tau \hat{f}(\tau), \partial_x \hat{f}(\tau), \partial_A \hat{f}(\tau) \right) \in \mathscr{C}^\alpha_{k-\left[\frac{n}{2}\right]-1}(\mathbf{H}_\tau) . \qquad (6.80)$$

when the prescribed data are as in Theorem 6.3.

PROOF: In order to apply the Gronwall-type Lemma 5.2 of [13], we need to prove that all the norms in $\hat{M}$ (see (6.54) and (6.58)) are controlled by the energy or the L^∞-norm of u. Since $k > \left[\frac{n}{2}\right] + 1$, from the weighted Sobolev's inequality, we have:

$$\|(\partial_\tau - \partial_x, \partial_x, \partial_A)\hat{f}\|^2_{\mathscr{B}^\alpha_1} \leq \|(\partial_\tau - \partial_x, \partial_x, \partial_A)\hat{f}\|^2_{\mathscr{H}^\alpha_k} \leq E^\alpha_{k,\lambda}[u(\tau)] . \quad (6.81)$$

Let us look at the L^∞-norm of $(\partial_\tau - \partial_x)\mathfrak{g}^\sharp$. Recall that the expression of $\mathfrak{g}^\sharp$ is given by (6.23). We estimate here only its worse term which is of the form $\Omega^{\frac{n-5}{2}}\widehat{H}$. We have:

$$
\begin{aligned}
\|(\partial_\tau - \partial_x)(\Omega^{\frac{n-5}{2}}\widehat{H})\|^2_{L^\infty} &\leq C\left(\|\Omega^{\frac{n-5}{2}}(\partial_\tau - \partial_x)\widehat{H}\|^2_{L^\infty} + \|\Omega^{\frac{n-7}{2}}\widehat{H}\|^2_{L^\infty} \right) \\
&\leq C\left(\|u\|^2_{L^\infty} + E^\alpha_{k,\lambda}[u(\tau)] \right) \qquad \text{for} \quad n \geq 7 .
\end{aligned}
$$

Thus,

$$\|(\partial_\tau - \partial_x)\mathfrak{g}^\sharp\| \leq C\left(\|u\|^2_{L^\infty} + E^\alpha_{k,\lambda}[u(\tau)] \right) \qquad \text{for} \quad n \geq 7 . \qquad (6.82)$$

Next, since $k > n/2 + 1$, as in (6.61), we have

$$
\begin{aligned}
\|\mathfrak{g}^\sharp\|^2_{\mathscr{C}^0_1} &\leq \|\mathfrak{g}^\sharp\|^2_{\mathscr{G}^0_k} \\
&\leq C\left(M_1 + E^\alpha_{k,\lambda}[u(\tau)] \right), \quad \text{for} \quad n \geq 5 . \qquad (6.83)
\end{aligned}
$$

Similarly,

$$\|\mathfrak{h}^\sharp\|^2_{\mathscr{C}^0_1} \leq \|\mathfrak{h}^\sharp\|^2_{\mathscr{G}^0_k} \leq C\left(M_1 + E^\alpha_{k,\lambda}[u(\tau)] \right) \quad \text{for} \quad n \geq 7 . \qquad (6.84)$$

If $\alpha = -1/2$ the threshold $n = 7$ in (6.82) and (6.84) can be lowered to $n = 6$ by using the estimate (6.73) on the slowest decaying term ψ.

To estimate the $\mathscr{C}^0_1$-norms of the harmonicity functions, we use again as in the previous estimate the Sobolev inequality and obtain a control of these norms by the energy with the same constrains as in (6.62)-(6.65). Let us estimate now the L^∞-norm of u. Integrating backward along the integral

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curve of the vector field $Y^\nu \partial_\nu = \partial_\tau - \partial_x$ we can write the identity (here we omit the variable v^A)

$$u(\tau, x) - u(\tau_0, \tau - \tau_0 + x) = \int_{\tau_0}^{\tau} (\partial_\tau - \partial_x)\, u(s, \tau - s + x)ds \ . \qquad (6.85)$$

Thus we have

$$
\begin{aligned}
|u(\tau, x)| &\leq |u(\tau_0, \tau - \tau_0 + x)| + \int_{\tau_0}^{\tau} \left| (\tau - s + x)^{-\alpha} (\partial_\tau - \partial_x)\, u(s, \tau - s + x) \right| (\tau - s + x)^\alpha\, ds \\
&\leq |u(\tau_0, \tau - \tau_0 + x)| + \int_{\tau_0}^{\tau} \| (\partial_\tau - \partial_x)\, u(s) \|_{\mathscr{C}_0^\alpha} (\tau - s + x)^\alpha\, ds \\
&\leq \|u(\tau_0)\|_{L^\infty} + \int_{\tau_0}^{\tau} \| (\partial_\tau - \partial_x)\, u(s) \|_{\mathscr{C}_0^\alpha} (\tau - s)^\alpha\, ds \ .
\end{aligned}
$$

Since $k > \frac{n}{2}$ we can now write ($-1 < \alpha \leq -1/2$):

$$
\begin{aligned}
\|u(\tau)\|_{L^\infty} &\leq \|u(\tau_0)\|_{L^\infty} + \int_{\tau_0}^{\tau} \| (\partial_\tau - \partial_x)\, u(s) \|_{\mathscr{H}_k^\alpha} (\tau - s)^\alpha\, ds \\
&\leq \|u(\tau_0)\|_{L^\infty} + \int_{\tau_0}^{\tau} \sqrt{E_{k,\lambda}^\alpha[u(s)]}\, (\tau - s)^\alpha\, ds \qquad (6.86)
\end{aligned}
$$

Inequalities (6.81)-(6.86) show that from (6.58) we have the following:

$$
\begin{aligned}
\|u(\tau)\|_{L^\infty}^2 + E_{k,\lambda}^\alpha[u(\tau)] &\leq C \left(\|u(\tau_0)\|_{L^\infty} + E_{k,\lambda}^\alpha[u(\tau_0)] \right) \\
&\quad + \int_{\tau_0}^{\tau} \Phi \left(E_{k,\lambda}^\alpha[u(s)],\ \|u(s)\|_{L^\infty} \right) (1 + (\tau - s)^\alpha)\, ds\ , \quad (6.87)
\end{aligned}
$$

where Φ is bounded on bounded sets. We can now apply Lemma 5.2 of [13] and obtain that there exists a time $\tau_0 < \tau_* < 0$ depending on $\|u(\tau_0)\|_{L^\infty} + E_{k,\lambda}^\alpha[u(\tau_0)]$ and on the function Φ such that $\forall \tau \in [\tau_0, \tau_*]$,

$$\|u(\tau)\|_{L^\infty} + E_{k,\lambda}^\alpha[u(\tau)] \leq 1 + C \left(\|u(\tau_0)\|_{L^\infty} + E_{k,\lambda}^\alpha[u(\tau_0)] \right) \ , \qquad (6.88)$$

which provides the desired bounds.

If one knows a priori that $\hat{M}(\tau)$ is bounded, (6.87) becomes effectively a linear inequality, and the claimed global bound immediately follows.

Actually, the solution constructed here is defined on $\mathscr{U}_{\tau_*}$ (see Figure 6.1). In order to obtain a solution in a whole neighborhood of the hyperboloid $\mathscr{S}_0$, we proceed as follows: Let $R > 0$ be a real positive number such that the level set $r = R$ lies in the region where the energy estimates above apply.

We consider the Cauchy problem for (6.13) with initial data obtained by restriction on

$$\mathscr{S}_0(R) = \mathscr{S}_0 \cap \{(x^\mu) : 0 \le |\vec{x}| \le R\} \,.$$

We thus obtain a Cauchy problem on a compact region. We can now apply to this problem the conclusion of Proposition 3.2, p. 378 of [24]: there exists a time $\tau_+ \in]\tau_0, 0[$ and a smooth solution on (see Figure 6.1)

$$\mathscr{V}_+ = \underset{t \in [t_0, -\frac{1}{2\tau_+}]}{\cup} \phi(\mathscr{S}_t(R)) \cap \mathscr{D}^+(\phi(\mathscr{S}_0(R))) \,,$$

where $\mathscr{D}^+$ denotes the domain of dependence, and where

$$\mathscr{S}_t(R) = \mathscr{S}_t \cap \{(x^\mu) : 0 \le |\vec{x}| \le R\} \,.$$

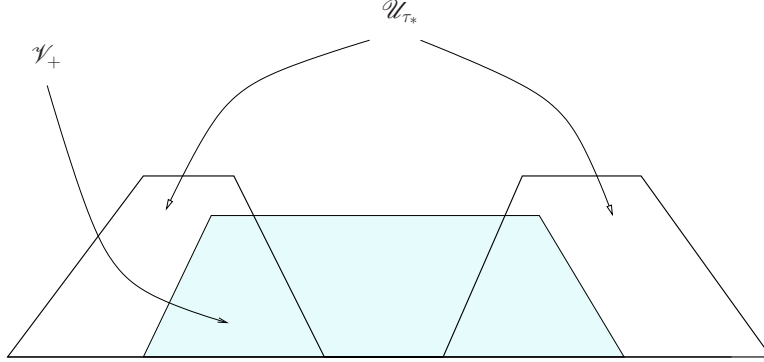From uniqueness in Proposition 3.2, p. 378 of [24], we conclude that the

Figure 6.1: The sets $\mathscr{V}_+$ and $\mathscr{U}_{\tau_*}$.

solutions constructed on $\mathscr{V}_+$ and $\mathscr{U}_{\tau_*}$ coincide on $\mathscr{V}_+ \cap \mathscr{U}_{\tau_*}$ which is not empty for R large enough. We thus obtain a solution of (6.13) with (6.51) in a whole neighborhood of $\mathscr{S}_0$. □

6.2.2 Space-regularity of the solution

For smooth initial data the solution constructed in the previous section is in $C^\infty(\mathscr{V}_+ \cup \mathscr{U}_{\tau_*})$. In this section we want to show that, for data given in the space $\underset{k \in \mathbb{N}}{\cap} \mathscr{H}_k^\alpha$, we can control the growth, near $x = 0$, of all space derivatives of the corresponding solution. We have the following:

THEOREM 6.5 *Under the hypotheses of Theorem 6.3, suppose moreover that the initial data given on the hyperboloid $\mathscr{S}_0$ satisfy*

$$\hat{f}\big|_{\phi(\mathscr{S}_0)} \in (\mathscr{H}_\infty^\alpha \cap L^\infty)\,(\mathbf{H}_{\tau_0}) \ \ \text{and}\ \ \partial \hat{f}\big|_{\phi(\mathscr{S}_0)} \in \mathscr{H}_\infty^\alpha(\mathbf{H}_{\tau_0}) \ . \qquad (6.89)$$

If $\alpha = -1/2$ and $n = 6$ we also suppose that (6.76) holds for all k. Let τ_* be as in Theorem 6.3 with $k = k_0$, where k_0 is the smallest integer larger than $[n/2] + 1$. Then

$$\forall \tau \in [\tau_0, \tau_*] \qquad \hat{f}(\tau) \in (\mathscr{H}_\infty^\alpha \cap L^\infty)(\mathbf{H}_\tau) \ , \quad \partial \hat{f}(\tau) \in \mathscr{H}_\infty^\alpha(\mathbf{H}_\tau) \ . \qquad (6.90)$$

Furthermore, any solution with smooth initial data as above for which $\hat{M}(\tau)$, as defined in (6.54), is bounded on $[\tau_0, \tau_1]$ satisfies (6.90) with $\tau_* = \tau_1$.

PROOF: We provide the details for $n > 6$; the treatment of the case $n = 6$ is similar. From Theorem 6.3 there exists a time τ_* and a constant C^* depending on k_0 such that $\forall \tau \in [\tau_0, \ \tau_*[$,

$$\|u(\tau)\|_{L^\infty}^2 + E_{k_0,\lambda}^\alpha[u(\tau)] \leq C^* \ . \qquad (6.91)$$

Now let $k \in \mathbb{N}$, $k \geq k_0$, since $\hat{f}\big|_{\phi(\mathscr{S}_0)} \in (\mathscr{H}_k^\alpha \cap L^\infty)\,(\mathbf{H}_{\tau_0})$ inequality (6.58) holds. Now the function $C_3(\hat{M}(s))$ appearing in this inequality is controlled by $E_{k_0,\lambda}^\alpha[u(\tau)]$ and thus by C^*, therefore, from (6.91) we have:

$$E_{k,\lambda}^\alpha[u(\tau)] \leq C(C^*) \left(1 + \int_{\tau_0}^\tau E_{k,\lambda}^\alpha[u(s)]ds \right) \ .$$

Applying Gronwall's inequality we obtain:

$$E_{k,\lambda}^\alpha[u(\tau)] \leq C e^{C\tau_*} \ .$$

This inequality shows that, for all k,

$$\partial u \in \mathscr{H}_k^\alpha \ , \qquad (6.92)$$

as desired. □

6.2.3 Estimates on time derivatives of the solution

In order to estimate the time derivatives of the solution, we introduce a new set of variables $(y, \tilde{x})$ (compare Figure 6.2):

$$\begin{cases} \tau = \frac{y - \tilde{x}}{2} + \tau_0 \\ x = \tilde{x} \end{cases} \quad \text{which implies that} \quad \begin{cases} \partial_y = \frac{1}{2}\partial_\tau \\ \partial_{\tilde{x}} = \partial_x - \frac{1}{2}\partial_\tau \end{cases} .$$
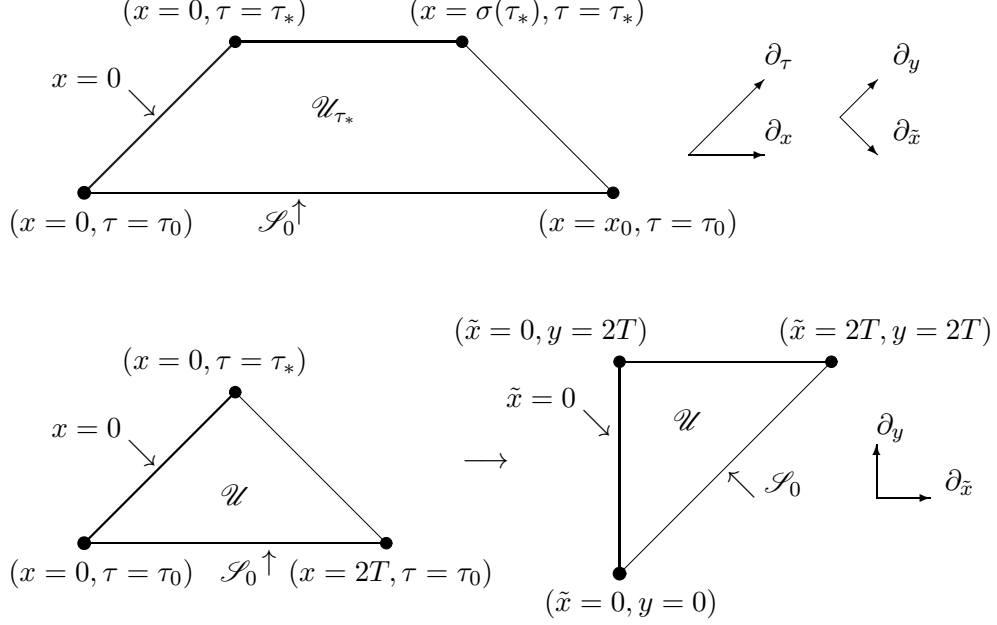
Figure 6.2: The variables (x, τ) and $(\tilde{x}, y)$, with $T := \tau_* - \tau_0$. The function σ has been introduced in (5.17). We hope that the reader will not get confused by the fact that the boundary $x = 0$, at the left-hand sides of the figures here, is depicted at the right-hand side of Figure 5.1.

Note that in these new coordinates, the hyperboloid $\mathscr{S}_0$ is represented by the set $\{y = \tilde{x}\}$. Since we are interested in the behavior of solution in a neighborhood of the set $\{x = 0\}$, as in [12] we restrict our attention on the subset $\mathscr{U}$ of $\mathscr{U}_{\tau_*}$ defined by:

$$\mathscr{U} = \left\{ (y, \tilde{x}, v^A) : 0 < x < y, \ v \in \mathscr{O}, 0 < y < 2(\tau_* - \tau_0) \right\} \ .$$

Recall that the definitions of the spaces

$$\mathscr{C}^\alpha_{\{x=0\},k}(\mathscr{U}), \quad \mathscr{C}^\sigma_{\{y=0\},k}(\mathscr{U}), \quad \mathscr{C}^\alpha_{\{0 \le x \le y\},k}(\mathscr{U}), \quad \text{and} \quad \mathscr{C}^{\alpha,\sigma}_{\{0 \le x \le y\},k}(\Omega) \ ,$$

can be found in Appendix A.1 page 116 of [12] with ∂_x there corresponding to $\partial_{\tilde{x}}$ here.

REMARK 6.6 In the coordinates $(y, \tilde{x})$ the components of the inverse of the

metric read (compare 6.39):

$$\mathfrak{g}^{yy} = 4\left(\mathfrak{g}^{\tau\tau} + \mathfrak{g}^{x\tau}\right) + \mathfrak{g}^{xx} = \mathrm{O}(x^{\frac{n-5}{2}}) \tag{6.93}$$

$$\mathfrak{g}^{y\tilde{x}} = 2\mathfrak{g}^{x\tau} + \mathfrak{g}^{xx} \tag{6.94}$$

$$\mathfrak{g}^{yA} = 2\mathfrak{g}^{\tau A} + \mathfrak{g}^{xA} \tag{6.95}$$

$$\mathfrak{g}^{\tilde{x}\tilde{x}} = \mathfrak{g}^{xx} = \mathrm{O}(x^{\frac{n-1}{2}}) \tag{6.96}$$

$$\mathfrak{g}^{\tilde{x}A} = \mathfrak{g}^{xA} . \tag{6.97}$$

Recall that the hypersurfaces $\mathscr{S}_s$ have been defined in (6.74). As a first step towards proving propagation of polyhomogeneity, we obtain some information about the ∂_y-derivatives of the fields:

THEOREM 6.7 *Suppose that* $k > \left[\frac{n}{2}\right] + 1$. *Under the hypotheses of Proposition 6.1, there exists* $t_* > t_0$ *and a solution of (6.13) defined on* $\underset{s\in[t_0,\, t_*]}{\cup} \mathscr{S}_s$ *such that:*

$$\hat{f} \in \left(\mathscr{C}^\alpha_{\{0\leq x\leq y\},k-\left[\frac{n}{2}\right]-1} \cap L^\infty\right)(\mathscr{U}) . \tag{6.98}$$

$$\left(\partial_\tau \hat{f}, \partial_x \hat{f}, \partial_A \hat{f}\right) \in \mathscr{C}^\alpha_{\{0\leq x\leq y\},k-\left[\frac{n}{2}\right]-1}(\mathscr{U}) , \tag{6.99}$$

where f *and* $\hat{f}$ *are defined by (6.51)-(6.52).*

PROOF: The proof of existence is given by Theorem 6.3 and we have $\hat{f} \in L^\infty(\mathscr{U})$, $\partial\hat{f} \in \mathscr{C}^\alpha_{\{\tilde{x}=0\},k-\left[\frac{n}{2}\right]-1}(\mathscr{U})$. We note that from (6.4) and (6.5) we have:

$$\Omega = \tilde{x}(-y - 2\tau_0), \qquad y\partial_y\Omega = -\tilde{x}y, \qquad \tilde{x}\partial_{\tilde{x}}\Omega = \Omega \qquad \text{and} \qquad \partial_A\Omega = 0 . \tag{6.100}$$

Identities (6.100) show that if we apply to (6.21) the operator $(\partial_A, \tilde{x}\partial_{\tilde{x}}, y\partial_y)$, then we obtain a wave equation with $(u, \partial_A u, y\partial_y u, \tilde{x}\partial_{\tilde{x}} u)$ as the new unknown functions in which the coefficients have the same powers of x as in the original equation, and the source term the same structure. More precisely, set

$$U = \begin{pmatrix} u \\ \partial_A u \\ \tilde{x}\partial_{\tilde{x}} u \\ y\partial_y u \end{pmatrix}, \qquad \text{we thus obtain} \qquad \begin{pmatrix} U \\ \partial U \end{pmatrix} = \begin{pmatrix} u \\ \partial_A u \\ \tilde{x}\partial_{\tilde{x}} u \\ y\partial_y u \\ \partial u \\ \partial(\partial_A u) \\ \partial(\tilde{x}\partial_{\tilde{x}} u) \\ \partial(y\partial_y u) \end{pmatrix}, \qquad (6.101)$$

and let us derive a wave equation on U. Straightforward calculations lead to the following identity (here we write the source term as a function of variables p_1 and p_2^σ, that is $\mathcal{F} = \mathcal{F}(\cdot,\, p_1, p_2^\sigma)$):

$$
\begin{aligned}
\Box_{\mathfrak{g}}(y\partial_y u) \;=\;& -(y\partial_y \mathfrak{g}^{\alpha\beta})\partial^2_{\alpha\beta}u + 2\mathfrak{g}^{\alpha y}\partial_\alpha\partial_y u - (y\partial_y \Upsilon^\alpha)\partial_\alpha u + \Upsilon^y \partial_y u \\
&+ (y\partial_y \mathcal{F})(\cdot, u, \partial u) + (y\partial_y u)\frac{\partial \mathcal{F}}{\partial p_1}(\cdot, u, \partial u) \\
&+ (\partial_y(y\partial_\sigma u) - \delta^y_\sigma \partial_y u)\,\frac{\partial \mathcal{F}}{\partial p_2^\sigma}(\cdot, u, \partial u)\ .
\end{aligned}
\tag{6.102}
$$

We write

$$
(y\partial_y \mathfrak{g}^{yy})\partial^2_y u = \partial_y \mathfrak{g}^{yy}\left(\partial_y(y\partial_y u) - \partial_y u\right) \sim \Omega^{\frac{n-5}{2}}(\partial U + U)\partial U\ ,
$$

$$
(y\partial_y \mathfrak{g}^{\tilde{x}y})\partial_y\partial_{\tilde{x}} u = \partial_y \mathfrak{g}^{yy}\partial_y(\tilde{x}\partial_{\tilde{x}} u) \sim \Omega^{\frac{n-5}{2}}\partial U \partial U\ ,
$$

$$
(y\partial_y \mathfrak{g}^{\tilde{x}\tilde{x}})\partial^2_{\tilde{x}} u = O(\tilde{x}^{\frac{n-3}{2}})\left(\partial_{\tilde{x}}(\tilde{x}\partial_{\tilde{x}} u) - \partial_{\tilde{x}} u\right) \sim \Omega^{\frac{n-3}{2}}U\partial U \quad \text{see (6.39)},
$$

$$
\mathfrak{g}^{yy}\partial^2_y u = O(\tilde{x}^{\frac{n-7}{2}})\frac{\tilde{x}}{y}\left(\partial_y(y\partial_y u) - \partial_y u\right) \sim \Omega^{\frac{n-7}{2}}U\partial U\ ,
$$

and

$$
2\mathfrak{g}^{\alpha y}\partial_\alpha\partial_y u = \mathfrak{g}^{yy}\partial^2_y u - \left\{\mathfrak{g}^{\tilde{x}\tilde{x}}\partial^2_{\tilde{x}} u + 2\mathfrak{g}^{\tilde{x}A}\partial_{\tilde{x}}\partial_A u + \mathfrak{g}^{AB}\partial_A\partial_B u + \Upsilon^\sigma\partial_\sigma u - \mathcal{F}(u, \partial u)\right\}\ .
$$

All the terms arising above have a structure similar to (6.50). A similar comparison of the remaining terms shows that we have

$$
\Box_{\mathfrak{g}}(y\partial_y u) = \mathcal{F}_1(U, \partial U)\ ,
\tag{6.103}
$$

where the source term $\mathcal{F}_1$ is of the general form as in (6.50) with the difference that it has a term $\Omega^{\frac{n-7}{2}}U\partial U$ with a multiplicative $\Omega^{\frac{n-7}{2}}$; this term can be estimated as in (6.68) as long as $n \geq 7$. Moreover, it is easily checked that this remains compatible with the estimate of Proposition 6.1 (see Remark 6.2). Note that the procedure above introduces into the coefficients of the source terms the function $(y, \tilde{x}) \longmapsto \frac{\tilde{x}}{y}$, which is bounded on $\mathscr{U}$; furthermore, $\tilde{x}\partial_{\tilde{x}}\frac{\tilde{x}}{y} = -y\partial_y\frac{\tilde{x}}{y} = \frac{\tilde{x}}{y}$, which implies that we will not loose the regularity of the source terms, as needed for the problem at hand, when iterating the process.

From the identities,

$$
\begin{aligned}
\Box_{\mathfrak{g}}(\tilde{x}\partial_{\tilde{x}} u) \;=\;& -(\tilde{x}\partial_{\tilde{x}} \mathfrak{g}^{\alpha\beta})\partial^2_{\alpha\beta}u + 2\mathfrak{g}^{\alpha\tilde{x}}\partial_\alpha\partial_{\tilde{x}} u - (\tilde{x}\partial_{\tilde{x}} \Upsilon^\alpha)\partial_\alpha u + \Upsilon^{\tilde{x}} \partial_{\tilde{x}} u \\
&+ (\tilde{x}\partial_{\tilde{x}} \mathcal{F})(\cdot, u, \partial u) + \partial_{\tilde{x}}(\tilde{x}\partial_{\tilde{x}} u)\frac{\partial \mathcal{F}}{\partial p_1}(\cdot, u, \partial u) \\
&+ \left(\partial_{\tilde{x}}(\tilde{x}\partial_\sigma u) - \delta^{\tilde{x}}_\sigma \partial_{\tilde{x}} u\right)\frac{\partial \mathcal{F}}{\partial p_2^\sigma}(\cdot, u, \partial u)\ ,
\end{aligned}
\tag{6.104}
$$

$$\Box_{\mathfrak{g}}(\partial_A u) = -(\partial_A \mathfrak{g}^{\alpha\beta})\partial^2_{\alpha\beta}u - (\partial_A \Upsilon^\alpha)\partial_\alpha u + \partial_A u \frac{\partial \mathcal{F}}{\partial p_1}(u, \partial u) + \partial\partial_A u \frac{\partial \mathcal{F}}{\partial p_2}(u, \partial u) ,$$
$$(6.105)$$

we deduce that the same analysis holds for $\Box_{\mathfrak{g}}(\partial_A u)$ and $\Box_{\mathfrak{g}}(\tilde{x}\partial_{\tilde{x}} u)$. Therefore we have derived for the new unknown function U a wave equation of the form (6.21), i.e.:

$$\Box_{\mathfrak{g}} U = \mathfrak{F}(U, \, \partial U) . \tag{6.106}$$

In order to apply to this equation Theorem 6.3, we have to check that the initial data for U are in the right spaces. Note that the initial data are prescribed on the subset $\{x = y\}$ of $\mathcal{U}$. We denote this hypersurface by Σ_0, thus $\Sigma_0 = \phi(\mathcal{S}_0) \cap \mathcal{U}$, and we set

$$\Sigma_s = \phi(\mathcal{S}_s) \cap \mathcal{U} \subset \mathbf{H}_{-1/2s} . \tag{6.107}$$

We want to prove the following.

LEMMA 6.8 *Under the hypotheses of Proposition 6.1 we have:*

$$(u, \partial_A u, \tilde{x}\partial_{\tilde{x}} u, y\partial_y u)|_{\Sigma_0} \in (\mathcal{H}^\alpha_k \cap L^\infty)(\Sigma_0) , \tag{6.108}$$

$$(\partial u, \partial\partial_A u, \partial(\tilde{x}\partial_{\tilde{x}} u), \partial(y\partial_y u))|_{\Sigma_0} \in \mathcal{H}^\alpha_{k-1}(\Sigma_0) . \tag{6.109}$$

PROOF: By assumption, we have

$$u|_{\Sigma_0} \in (\mathcal{H}^\alpha_k \cap L^\infty)(\Sigma_0), \quad \text{and} \quad (\partial_A u, \partial_{\tilde{x}} u \,, \partial_y u)|_{\Sigma_0} \in \mathcal{H}^\alpha_k(\Sigma_0) . \tag{6.110}$$

Now, using Sobolev's embedding theorem, we have

$$\tilde{x}^{-\alpha} (\partial_A u, \partial_{\tilde{x}} u \,, \partial_y u)\big|_{\Sigma_0} \in L^\infty(\Sigma_0) . \tag{6.111}$$

This leads to the following estimates:

$$\begin{aligned}
|\tilde{x}\partial_{\tilde{x}} u|_{\Sigma_0}| &= \tilde{x}^{1+\alpha} \left|\tilde{x}^{-\alpha}\partial_{\tilde{x}} u|_{\Sigma_0}\right| < \infty, \\
|y\partial_y u|_{\Sigma_0}| &= |\tilde{x}\partial_y u|_{\Sigma_0}| = \tilde{x}^{1+\alpha} \left|\tilde{x}^{-\alpha}\partial_y u|_{\Sigma_0}\right| < \infty .
\end{aligned}$$

To see that $\partial_A u(\tau_0)$ is in $L^\infty(\mathcal{S}_0)$, we proceed as follows: integrating $\partial_A u(\tau_0)$ in x until x_0 gives the inequality

$$\partial_A u(\tau_0, x_0, v^A) - \partial_A u(\tau_0, \tilde{x}, v^A) = \int_{\tilde{x}}^{x_0} \partial_{\tilde{x}} \partial_A u(\tau_0, s, v^A)ds ,$$

which leads to the estimate

$$\begin{aligned}
|\partial_A u(\tau_0, \tilde{x}, v^A)| &\leq |\partial_A u(\tau_0, x_0, v^A)| + \|\partial_{\tilde{x}} u(\tau_0)\|_{\mathscr{C}^\alpha_{\{\tilde{x}=0\},1}} \int_{\tilde{x}}^{x_0} s^\alpha ds \\
&\leq |\partial_A u(\tau_0, x_0, v^A)| + \|\partial_{\tilde{x}} u(\tau_0)\|_{\mathcal{H}^\alpha_k} \int_{\tilde{x}}^{x_0} s^\alpha ds
\end{aligned}$$

(recall $k - 1 > \frac{n}{2}$). Since

$$\|\partial_A u(\tau_0, x_0, v^A)\|_{L^\infty(\mathcal{O})} < \infty, \qquad \|\partial_{\tilde{x}} u(\tau_0)\|_{\mathscr{H}_k^\alpha} \leq E_k^\alpha[u(\tau_0)] < \infty ,$$

and

$$\int_{\tilde{x}}^{x_0} s^\alpha ds = \frac{1}{\alpha + 1} \left(x_0^{\alpha+1} - \tilde{x}^{\alpha+1} \right) < \infty ,$$

we conclude that $\|\partial_A u(\tau_0)\|_{L^\infty} < \infty$. Thus $(\partial_A u)\,|_{\Sigma_0} \in L^\infty(\Sigma_0)$ and we then obtain (6.108). On the other hand we have

$$
\begin{aligned}
\|\partial_\nu(\tilde{x}\partial_{\tilde{x}} u)|_{\Sigma_0}\|_{\mathscr{H}_{k-1}^\alpha(\Sigma_0)} &\leq \|\tilde{x}\partial_{\tilde{x}}(\partial_\nu u)|_{\Sigma_0}\|_{\mathscr{H}_{k-1}^\alpha(\Sigma_0)} + \|\delta_\nu^{\tilde{x}}\partial_{\tilde{x}} u|_{\Sigma_0}\|_{\mathscr{H}_{k-1}^\alpha(\Sigma_0)} \\
&\leq \|\partial_\nu u|_{\Sigma_0}\|_{\mathscr{H}_k^\alpha(\Sigma_0)} \\
&< \infty \qquad \text{see (6.110)} .
\end{aligned}
$$

Similarly, we have $\partial(y\partial_y u)|_{\Sigma_0}$, $\partial\partial_A u|_{\Sigma_0} \in \mathscr{H}_{k-1}^\alpha(\Sigma_0)$. We thus obtain (6.108) and the proof of the lemma is complete. $\qquad\square$

Now, we apply Theorem 6.3 to (6.106) and obtain that

$$(u, \partial_A u, \tilde{x}\partial_{\tilde{x}} u, y\partial_y u) \in L^\infty\left([0, 2(\tau_* - \tau_0)], \left(\mathscr{H}_k^\alpha \cap L^\infty\right)(\Sigma_\epsilon)\right), \qquad (6.112)$$

$$(\partial u, \partial\partial_A u, \partial(\tilde{x}\partial_{\tilde{x}} u), \partial(y\partial_y u)) \in L^\infty\left([0, 2(\tau_* - \tau_0)], \mathscr{H}_k^\alpha(\Sigma_\epsilon)\right). \qquad (6.113)$$

Using once more the Sobolev embedding theorem, we obtain that $\forall \epsilon \in [0, 2(\tau_* - \tau_0)]$

$$(u, \partial_A u, \tilde{x}\partial_{\tilde{x}} u, y\partial_y u)|_{\Sigma_\epsilon} \in \mathscr{C}_{\{\tilde{x}=0\},k-\left[\frac{n}{2}\right]-1}^\alpha(\Sigma_\epsilon) .$$

$$
\begin{aligned}
\|(u, \partial_A u, \tilde{x}\partial_{\tilde{x}} u, y\partial_y u)\|_{L^\infty(\mathscr{U})} &= \sup_{\tau\in[\tau_0,\tau_*]} \|(u, \partial_A u, \tilde{x}\partial_{\tilde{x}} u, y\partial_y u)|_{\mathscr{S}_\tau}\|_{L^\infty(\mathscr{S}_\tau)} \\
&\leq \sup_{\tau\in[\tau_0,\tau_*]} \|(u, \partial_A u, \tilde{x}\partial_{\tilde{x}} u, y\partial_y u)|_{\mathscr{S}_\tau}\|_{\mathscr{H}_k^\alpha(\mathscr{S}_\tau)} \\
&\underbrace{<}_{\text{see (6.112)}} \infty .
\end{aligned}
$$

Using now (6.113) instead of (6.112) we have

$$\|(u, \partial_A\partial u, \tilde{x}\partial_{\tilde{x}}\partial u, y\partial_y\partial u)\|_{L^\infty(\mathscr{U})} < \infty .$$

This allows us to conclude that $(u, \partial u)$ is in $\mathscr{C}_{\{0\leq\tilde{x}\leq y\},1}^\alpha(\mathscr{U})$. Now, if we repeat this process j times with $j = k - \left[\frac{n}{2}\right] - 1$ then we obtain that u is in $\mathscr{C}_{\{0\leq\tilde{x}\leq y\},k-\frac{n}{2}-1}^\alpha(\mathscr{U})$. This completes the proof of Theorem 6.7. $\qquad\square$

COROLLARY 6.9 *Under the hypotheses of Theorem 6.5 we have the following:*

$$\hat{f} \in \left(\mathscr{C}^{\alpha}_{\{0 \le \tilde{x} \le y\},\infty}(\mathscr{U}) \cap L^{\infty}\right)(\mathscr{U}) \qquad and \qquad \partial \hat{f} \in \mathscr{C}^{\alpha}_{\{0 \le \tilde{x} \le y\},\infty}(\mathscr{U}) .$$

PROOF: The result is a combination of Theorems 6.5 and 6.7. □

7 Polyhomogeneous solutions of the Einstein-Maxwell equations

Let δ be a positive real number. We recall that the spaces of polyhomogeneous functions $\mathscr{A}_{\{x=0\}}$, $\mathscr{A}^{\delta}_{\{x=0\}}$, $\mathscr{A}_{\{0 \le x \le y\}}$ and $\mathscr{A}^{\delta}_{\{0 \le x \le y\}}$ are defined in [12, Equations (A.1)-(A.2)]. We consider the Cauchy problem for the Einstein-Maxwell equations (6.13) with (6.51) in wave coordinates (x^{μ}) and Lorenz gauge with prescribed data on the hyperboloid $\mathscr{S}_0$ (see (6.56)) at the interior of the future light-cone with vertex the origin of coordinates. The coordinate x in which the polyhomogeneous expansion is taken is $x = \frac{1}{t+r}$ where $t = x^0$ and $r = |\vec{x}| = \sum\limits_{i=1}^{n}(x^i)^2)$. Indeed we have (see (6.4)):

$$
\begin{aligned}
x = -\tau - \rho &= -\frac{t}{-t^2 + r^2} - \left(\sum \frac{(x^i)^2}{(-t^2 + r^2)^2}\right)^{1/2} \\
&= -\frac{t}{-t^2 + r^2} - \frac{r}{t^2 - r^2} \\
&= \frac{1}{t+r} .
\end{aligned}
$$

We want to prove that, polyhomogeneous initial data for the above Cauchy problem lead to polyhomogeneous solution. We have the following:

THEOREM 7.1 *Consider the Einstein-Maxwell equations on $\mathbb{R}^{1+n}$, $n \ge 8$. Let $\delta \in \mathbb{R}$ be such that $1/(2\delta) \in \mathbb{N}$ when n is even and $1/\delta \in \mathbb{N}$ when n is odd. Suppose that the initial data for (6.13) in wave coordinates and Lorenz gauge are polyhomogeneous on the hyperboloid $\mathscr{S}_0$:*

$$f\big|_{\mathscr{S}_0} \in x^{\frac{n-1}{2}} \mathscr{A}^{\delta}_{\{x=0\}} \cap L^{\infty}, \qquad \partial_{\tau} f\big|_{\mathscr{S}_0} \in x^{\frac{n-1}{2}} \mathscr{A}^{\delta}_{\{x=0\}} , \qquad (7.1)$$

with $f = (g_{\mu\nu} - \eta_{\mu\nu}, A_{\mu})$. There exists a time $t_+ > t_0$ and a solution defined on $\bigcup\limits_{t \in [t_0, t_+]} \mathscr{S}_t$ such that $\forall t \in [t_0, t_+]$ we have:

$$f(t) = f\big|_{\mathscr{S}_t} \in x^{\frac{n-1}{2}} \mathscr{A}^{\delta}_{\{x=0\}} \qquad and \qquad \partial_{\tau} f(t) = \partial_{\tau} f\big|_{\mathscr{S}_t} \in x^{\frac{n-1}{2}-1} \mathscr{A}^{\delta}_{\{x=0\}} . \tag{7.2}$$

Moreover, the solution is polyhomogeneous at $\mathscr{I}$, in the above polyhomogeneity class, as long as it remains in $\mathscr{H}_k^\alpha(\mathbf{H}_\tau)$, for some $\alpha \in (-1, -1/2]$.

PROOF: Choose any $\alpha < 0$; we then have the inclusion $\mathscr{A}^\delta_{\{x=0\}}(\overline{\phi(\mathscr{S}_0)}) \subset \mathscr{H}_\infty^\alpha(\overline{\phi(\mathscr{S}_0)})$. It follows from (7.1) that we have:

$$\hat{f}\big|_{\phi(\mathscr{S}_0)} \in (\mathscr{H}_\infty^\alpha \cap L^\infty)(\phi(\mathscr{S}_0)) \quad \text{and} \quad \partial\hat{f}\big|_{\phi(\mathscr{S}_0)} \in \mathscr{H}_\infty^\alpha(\phi(\mathscr{S}_0)) \ . \quad (7.3)$$

For definiteness set $\alpha = -1/2$. From Theorem 6.5, there exists a time τ_* and a smooth solution $\hat{f}$ of (6.13)-(6.51)-(7.3) defined on $\mathscr{U}_{\tau_*}$ such that $\forall \tau \in [\tau_0, \tau_*]$, $\hat{f}(\tau) \in \mathscr{C}_j^\alpha(\mathbf{H}_\tau)$. Next, applying Corollary 6.9 one obtains that

$$\hat{f} \in \left(\mathscr{C}_{\{0 \le x \le y\}, \infty}^\alpha \cap L^\infty\right)(\mathscr{U}) \quad \text{and} \quad \partial\hat{f} \in \mathscr{C}_{\{0 \le x \le y\}, \infty}^\alpha(\mathscr{U}) \ .$$

From Theorem 3.1 of Section 3, with

$$\psi_1 = \hat{f} \ , \quad \psi_2 = (\partial_y \hat{f}, \partial_A \hat{f}) \ , \quad \varphi = \partial_x f \ ,$$

we obtain (7.2), and the proof is completed. $\qquad\square$

It is natural to find conditions which guarantee that solutions remain in weighted Sobolev spaces on hyperboloids, and hence remain polyhomogeneous if the initial data are. One such criterion is provided by the following:

THEOREM 7.2 *Suppose that $k > \left[\frac{n}{2}\right] + 1$, with $n = 6$ and $\alpha = -1/2$, or $n \ge 7$ with $\alpha \in (-1, 1/2]$. Solutions of the Einstein-Maxwell equations remain in $\mathscr{H}_k^\alpha$, $\alpha \in (-1, -1/2]$ as long as $\hat{f}$ remains in $\mathscr{C}_{\{x=0\}, 1}^\kappa$, with*

$$\kappa > -\frac{(n-7)}{2} \ . \quad (7.4)$$

The same is true for

$$\kappa > -\frac{(n-5)}{2} \text{ provided that } \|x^{\frac{n-7}{2}} y_\mu y_\nu \hat{H}^{\mu\nu}(\tau_0)\|_{L^\infty} < \infty \ . \quad (7.5)$$

In particular, in dimensions $n+1 \ge 9$ the small data solutions of [18, 19] evolving out from data stationary outside of a compact set are polyhomogeneous.

PROOF: We want to use Proposition 5.11 to show that solutions as above remain in $\mathscr{H}_k^\alpha$, $\alpha \in (-1, -1/2]$. For this, consider first the right-hand side of (5.94). For $\kappa \ge -(n-5)/2$ one immediately finds that $\|\delta\mathfrak{g}^\sharp\|_{\mathscr{C}_{\{x=0\}, 1}^0}$ is finite,

similarly for $(\partial_x - \partial_\tau)\delta\mathfrak{g}^\sharp$ when $\kappa \geq -(n-7)/2$. Finiteness of $\|\delta\mathfrak{h}^\sharp\|_{\mathscr{C}^0_{\{x=0\},1}}$ is straightforward for $\kappa \geq -(n-7)/2$ from (6.35)-(6.38). The estimate on $\delta\Upsilon$ follows from (6.46) and (6.48) provided again that $\kappa \geq (n-7)/2$.

For $\kappa \geq -(n-5)/2$ the slowest decaying terms in $\mathfrak{h}$, Υ, and in $(\partial_x - \partial_\tau)\mathfrak{g}^\sharp$ are handled by the $\mathscr{C}^0_{\{x=0\},1}$-spaces equivalent of (6.73),

$$
\|x^{\frac{(n-7)}{2}}\psi(\tau)\|_{\mathscr{C}^0_{\{x=0\},1}}
$$
$$
\leq \quad \|x^{\frac{(n-7)}{2}}\psi(\tau_0)\|_{\mathscr{C}^0_{\{x=0\},1}} + C(\tau_0,\tau_1)\int_{\tau_0}^\tau \|x^{\frac{(n-7)}{2}}\zeta(s)\|_{\mathscr{C}^0_{\{x=0\},1}}\,ds \ , \quad (7.6)
$$

under the supplementary condition that $\|x^{\frac{(n-7)}{2}}\psi(\tau_0)\|_{\mathscr{C}^0_{\{x=0\},1}}$ is finite.

For any σ such that

$$
\sigma < \kappa \tag{7.7}
$$

we have

$$
\mathscr{C}^\kappa_{\{x=0\},1} \subset \mathscr{B}^\sigma_1 \ .
$$

Hence the right-hand side of (5.95) is finite for all such σ's, and so (5.96) applies. It remains to show that the integrand in the second line of (5.96) can be bounded by a multiple of the energy:

$$
\|(\delta\mathfrak{g}^\sharp, \delta\mathfrak{h}^\sharp, \delta\Upsilon)\|^2_{\mathscr{G}^{\alpha-\sigma}_k(\mathbf{H}_\tau)} \leq C E^\alpha_k[u(s)] \ .
$$

This is easily checked to hold under (7.4) or (7.5) if we choose σ so that

$$
\sigma > -\frac{n-7}{2} \ .
$$

This, together with (7.7), explains (7.4).

The property that the solutions of the Einstein-Maxwell equations constructed by Loizelet are in $\mathscr{C}^\kappa_{\{x=0\},1}$ on all hyperboloidal slices has been verified in (4.22). There $-\kappa = \delta \in (0, 1/4)$ $\qquad\square$

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